$Lx = b$ where L is lower triangular

$l_{11} x_1 = b_1$

$l_{21} x_1 + l_{22} x_2 = b_2$

$- - - .$

$l_{n1} x_1 + l_{n2} x_2 + \dots + l_{nn} x_n = b_n$

Then solve for $x_1, x_2, \dots, x_n$: forward elimination

The operation count is again $\sim \frac{n^2}{2}$.

Gaussian elimination

Idea: use elementary operations to reduce $Ax = b$ to the upper triangular matrix and use back substitution to find x

1. multiply an equation by a nonzero constant and subtract from another equation

2. interchange two equations

Ex

$$\left(\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$ augmented matrix

Step 1 Eliminate variable x_1 from eq$^{\underline{ns}}$ 2 and 3.

$m_{21} = \dfrac{a_{21}}{a_{11}}$ $\quad a_{22} \leftarrow a_{22} - m_{21} \cdot a_{12}$

$a_{23} \leftarrow a_{23} - m_{21} \cdot a_{13}$

$b_2 \leftarrow b_2 - m_{21} \cdot b_1$

$m_{31} = a_{31} / a_{11}$ $\quad a_{32} \leftarrow a_{32} - m_{31}\, a_{12}$

$a_{33} \leftarrow a_{33} - m_{31} \cdot a_{13}$

$b_3 \leftarrow b_3 - m_{31} \cdot b_1$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{pmatrix}$$ These elements have been chaged

Step 2 Eliminate variable x_2 from equation 3

$m_{32} = a_{32} / a_{22}$ $\quad a_{33} \leftarrow a_{33} - m_{32} \cdot a_{23}$

$b_3 \leftarrow b_3 - m_{32} \cdot b_2$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & \boxed{a_{33}} & \boxed{b_3} \end{pmatrix}$$ These elements have been changed

Now we can use back substitution to find solution.

Ex

$$\begin{aligned} 2x_1 - x_2 \quad &= 1 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$2 \leftarrow 2 - \left(-\frac{1}{2}\right)(-1) = \frac{3}{2}$

$-1 \leftarrow -1 - \left(-\frac{1}{2}\right) \cdot 0 = -1$

$0 \leftarrow 0 - \left(-\frac{1}{2}\right) \cdot 1 = \frac{1}{2}$

$m_{21} = {}^{-1}/2 = -\frac{1}{2}$

$m_{31} = {}^{0}/2 = 0$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 3/2 & -1 & \frac{1}{2} \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$m_{32} = {-1}\Big/{\tfrac{3}{2}} = -\frac{2}{3}$$

$$2 \leftarrow 2 - \left(-\tfrac{2}{3}\right)(-1) = 4/3$$

$$1 \leftarrow 1 - \left(-\tfrac{2}{3}\right)\tfrac{1}{2} = 4/3$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 3/2 & -1 & 1/2 \\ 0 & 0 & 4/3 & 4/3 \end{array}\right)$$

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ +\tfrac{3}{2}x_2 - x_3 &= \tfrac{1}{2} \\ \tfrac{4}{3}x_3 &= \tfrac{4}{3} \end{aligned} \quad \uparrow$$

$$x_3 = \frac{4}{3} \Big/ \frac{4}{3} = 1$$

$$x_2 = \left(\frac{1}{2} - (-1)\right) \Big/ \frac{3}{2} = 1$$

$$x_1 = \left(1 - (-1 \cdot 1 + 0 \cdot 1)\right) / 2 = 1$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

General case

$$\left(A^{(k)} : b^{(k)}\right) = \left(\begin{array}{cccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & & & & b_1^{(1)} \\ & a_{22}^{(2)} & & & & & b_2^{(2)} \\ & & \ddots & & & & \vdots \\ & & & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} & b_k^{(k)} \\ & & & \vdots & & \vdots & \vdots \\ & & & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} & b_n^{(k)} \end{array}\right)$$

Step k: eliminate variable x_k from equations $k+1, .., n$

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \qquad i = k+1, .., n$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} \cdot a_{kj}^{(k)}, \quad i, j = k+1, .., n$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik} \cdot b_k^{(k)}, \quad i = k+1, .., n$$

m_{ik}: multiplier

$a_{kk}^{(k)}$: pivot

After the last step, pivot elements will be on the main diagonal.

code

```
% reduction to upper triangular form
%
for k=1:n-1
    for i = k+1:n
        xm = a(i,k) / a(k,k)
        for j = k+1:n
            a(i,j) = a(i,j) - xm * a(k,j)
        end
        b(i) = b(i) - xm* b(k)
    end
end
%
% back substitution
%
x(n) = b(n) / a(n,n)
for i=n-1:-1:1
    s=0;
    for j = i+1:n
        s = s + a(i,j) * x(j)
    end
    x(i) = (b(i) - s) / a(i,i)
end
```

Note: $a_{11}x_1 + \overbrace{a_{12}\underline{x_2} + \dots + a_{1n}\underline{x_n}}^{S} = b_1$

Operation count (in reduct. to upper triang.)

$\#$ divisions $= n(n-1)/2$

$\#$ multiplications $= n(n-1)(2n-1)/6$

Pf $\quad \#$ divisions $= \sum_{k=1}^{n-1} (n-k) = \sum_{k=1}^{n-1} k = n(n-1)/2$ as before

$\#$ multipl. $= \sum_{k=1}^{n-1} (n-k)^2 = \sum_{k=1}^{n-1} k^2 = S$