2/5/2010

Operation count (for reduction to upper triangular matrix

\# divisions $= n(n-1)/2$

\# multiplic. $= n(n-1)(2n-1)/6$

Proof for # multipl.

$$\# \text{mult} = \sum_{k=1}^{n-1} (n-k)^2 = \sum_{k=1}^{n-1} k^2 = S$$

$$(n-1)^3 = (n-1)^3 - (n-2)^3 + (n-2)^3 + \dots - 2^3 + 2^3 - 1^3 + 1^3$$

$$= \sum_{k=1}^{n-1} \left(k^3 - (k-1)^3\right) = \sum_{k=1}^{n-1} \left(\cancel{k^3} - (\cancel{k^3} - 3k^2 + 3k - 1)\right)$$

$$= +3 \underbrace{\sum_{k=1}^{n-1} k^2}_{\parallel \atop S} - 3 \underbrace{\sum_{k=1}^{n-1} k}_{\frac{n(n-1)}{2}} + \underbrace{\sum_{k=1}^{n-1} 1}_{n-1}$$

$$\Rightarrow (n-1)^3 = 3S - \frac{3}{2} n(n-1) + (n-1)$$

$$3S = +(n-1)^3 + \frac{3}{2} n(n-1) - (n-1)$$

$$3S = (n-1)\left((n-1)^2 + \frac{3}{2} n - 1\right) = (n-1)\left(n^2 - 2n + \cancel{1} + \frac{3}{2} n - \cancel{1}\right)$$

$$3S = (n-1)\left(n^2 - \frac{n}{2}\right) = (n-1)\frac{n}{2}(2n-1)$$

$$\Rightarrow \quad S = \frac{n(n-1)(2n-1)}{6} \quad \blacksquare$$

Note The leading order in operation count for Gaussian elimination is $\frac{n^3}{3}$.

Matrix form

Let $A^{(k)}$ be the reduced matrix at step k.
The effect of performing step k is

$$A^{(k+1)} = E_k A^{(k)}$$

where

$$E_k = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & -m_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ 0 & & -m_{nk} & & & 1 \end{pmatrix} \quad : \text{lower triangular matrix}$$

row $k+1$, column k

$$\text{row } k+1 \leftarrow \text{row}_{k+1} - m_{k+1,k} \cdot \text{row } k$$

Then $Ax = b$ is reduced to $A^{(n)} x = b^{(n)}$

where

$$A^{(n)} = E_{n-1} \dots E_2 E_1 A \quad : \text{upper triangular matrix}$$

$$b^{(n)} = E_{n-1} \dots E_2 E_1 b$$

LU factorization

Assume : $E_{n-1} \cdots E_2 E_1 A = U$: upper triangular

$(a_{kk}^{(k)} \neq 0)$

$$E_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & -m_{k+1,k} & 1 & & \\ & 0 & \vdots & & \ddots & \\ & & -m_{nk} & & & 1 \end{pmatrix} = I - m_k e_k^T$$

where $m_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix}$ $\qquad e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ k row

unit vector w/ 1
on k^{th} ~~pos~~ position
and 0's elsewhere

Ex 3x3 case

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{pmatrix}$$

where
$$\begin{pmatrix} 0 & 0 & 0 \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ m_{21} \\ m_{31} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Since $U = E_{n-1} \dots E_2 E_1 A$,

$$A = (E_{n-1} \dots E_2 E_1)^{-1} U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \dots E_{n-1}^{-1}} \cdot U$$

1. $$E_k^{-1} = (I - m_k e_k^T)^{-1} = I + m_k e_k^T$$

Pf
$$(I - m_k e_k^T)(I + m_k e_k^T) = I + \cancel{m_k e_k^T} + \cancel{- m_k e_k^T} - m_k \underbrace{e_k^T m_k}_{0} e_k^T$$

$e_k^T m_k = 0$ (inner product)

$$e_k^T m_k = \begin{array}{c} \scriptstyle k \text{ column} \\ \downarrow \\ \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \end{pmatrix} \end{array} \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix} & \begin{array}{l} \\ \\ k \text{ row} \\ k+1 \text{ row} \\ \\ \\ \end{array} \end{array} = 0$$

2. $$\cancel{E_1^{-1} E_2^{-1} = (I + m_1 e_1^T)(I + m_2 e_2^T)}$$

$$E_1^{-1} E_2^{-1} = I + m_1 e_1^T + m_2 e_2^T$$

Proof $E_1^{-1}E_2^{-1} = (I + m_1 e_1^T)(I + m_2 e_2^T) =$

$$= I + m_2 e_2^T + m_1 e_1^T + \cancelto{0}{m_1 \underbrace{e_1^T m_2}_{0} e_2^T}$$

$$\underset{m_2}{e_1^T} = (1\ 0\ \dots\ 0)\begin{pmatrix} 0 \\ 0 \\ m_{32} \\ \vdots \\ m_{n2} \end{pmatrix} = 0$$

i.e. $e_1^T m_2 = 0$

thus

$$E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & & & & 0 \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & 0 & & 1 \end{pmatrix}$$

and ~~thus~~

3. $$E_1^{-1}E_2^{-1}\dots E_{n-1}^{-1} = \begin{pmatrix} 1 & & & & 0 \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & \dots & m_{n,n-1} & 1 \end{pmatrix}$$

$$E_1^{-1}E_2^{-1}\dots E_{n-1}^{-1} = I + m_1 e_1^T + m_2 e_2^T + \dots + m_{n-1} e_{n-1}^T$$

$\equiv L$: lower triangular matrix

Note Then $A = LU$: L: unit lower triangular

U: upper triangular

$$A = \underbrace{E_1^{-1}E_2^{-1}\dots E_{n-1}^{-1}}_{L} U$$

Application

$Ax=b$

$\underbrace{LU}_{y}x=b$

Solve $Ax=b$

1. Find L, U such that $LU=A$ $\quad O(n^3)$
2. Solve $Ly=b$
3. Solve $Ux=y$

$\left.\begin{array}{l} \\ \end{array}\right\}$ triangular systems $\quad O(n^2)$

Then $Ax = L\underbrace{Ux}_{y} = Ly = b$

Advantage: If you solve for L and U, then you can solve for many vectors b using steps 2 and 3.