2/8/2010

<u>Newton's method:</u> $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ given x_0

if root α is simple that $|\alpha - x_{n+1}| \le C|\alpha - x_n|^2$

2nd order convergence

if root α is multiple w/ multiplicity m then

$$|\alpha - x_{n+1}| \le C|\alpha - x_n|$$

How to restore 2nd order convergence?

I $\quad x_{n+1} = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}$

where m is multiplicity of root α.

II Assume that α is a root of $f(x)$ multiplicity m.
Then we can write

$$f(x) = (x-\alpha)^m h(x)$$

where
$h(\alpha) \neq 0$

$$f'(x) = m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)$$

$$= (x-\alpha)^{m-1}\left[m h(x) + (x-\alpha) h'(x)\right]$$

Define $F(x) = \frac{f(x)}{f'(x)} = \frac{(x-\alpha)^m h(x)}{(x-\alpha)^{m-1}[m h(x) + (x-\alpha) h'(x)]}$

$$F(x) = (x-\alpha) \frac{h(x)}{m h(x) + (x-\alpha) h'(x)}$$

$$F(\alpha) = 0 \cdot \frac{h(\alpha)}{m h(\alpha) + 0 \cdot h'(\alpha)} = 0$$

$\Rightarrow \alpha$ is a root of F

Differentiate $F(x)$. We can show that $F'(\alpha) \neq 0$
$\Rightarrow \alpha$ is a single root of F. Therefore, we can use Newton's method applied to function F and thus we can compute root of α with 2nd order scheme.

Summary given $f(x) = 0$ α is a root of f

Consider $F(x) = \frac{f(x)}{f'(x)}$

Apply Newton's method to $F(x)$.

Secant method (restored convergence)

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \Rightarrow x_{n+1} = x_n - m \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

where m is multiplicity of the root.

Back to direct methods.

Recall

LU decomposition

Given $Ax=b$ $\qquad\qquad$ $L\underbrace{Ux}_{y}=b$

1. $A=LU$ $\qquad$ $O\left(\frac{n^3}{3}\right)$

2. Solve $Ly=b$ for y $\qquad$ $O\left(\frac{n^2}{2}\right)$

3. Solve $Ux=y$ for x $\qquad$ $O\left(\frac{n^2}{2}\right)$

Ex

$$A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We found previously: $m_{21}=-1/2$

$m_{31}=0$ $\qquad$ $m_{32}=-2/3$

We also reduced A to the upper triangular matrix:

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}=U$$

$$A = LU$$

Therefore, we can write

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}}_{U}$$

Solve $Ax = b$, $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Solve $Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} y_1 = 1 \\ y_2 = \frac{1}{2} \\ y_3 = 4/3 \end{matrix}$$

$-\frac{1}{2} \cdot y_1 + y_2 = 0$

$-\frac{2}{3} y_2 + y_3 = 1$ $\Rightarrow y = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{pmatrix}$

$y_3 = 1 + \frac{2}{3} \cdot \frac{1}{2}$

Solve $Ux = y$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 4/3 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{matrix}$$

$\frac{3}{2} x_2 - x_3 = 1/2$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Storage (triangularazation)

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} \\ m_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ m_{n1} & & m_{n,n-1} & a_{nn}^{(n)} \end{pmatrix}$$

Pivoting

It may happen that some pivot elements are zero even though matrix A is nonsingular.
If this is the case, then the above procedure breaks down.

Ex

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}}_{A}\begin{matrix} \vdots & 1 \\ \vdots & 2 \\ \vdots & 1 \end{matrix}$$

$$\det A = 1\cdot(-1)^{2+1}\cdot\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} +$$

$$+2(-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1\cdot\cancelto{0}{(2-2)} + 1\cdot(2-1) - 2(2-1)$$

$$= -1$$

$\det A \neq -1 \Rightarrow A$ is nonsingular

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right)$$

$m_{21} = 1/1 = 1$

$m_{31} = 1/1 = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$a_{22}^{(2)} = 0$ even though $\det A = -1 \neq 0$

$m_{32} = 1/0$

<u>Remedy</u>: interchange rows 2 and 3 and proceed.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$x_3 = 1/1 = 1$

$x_2 = -1 = (0 - 1 \cdot \overset{=1}{x_3})/1$

$x_1 = (1 - (1 \cdot \underset{-1}{\underset{\shortparallel}{x_2}} + 1 \cdot \underset{1}{\underset{\shortparallel}{x_3}}))/1 = 1$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$