3/1/2010

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$$

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

Last time we computed two linearly independent e'vectors $\vec{p}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{p}_2 = \begin{pmatrix}2\\1\end{pmatrix}$

$$e_0 = x - x_0 = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \vec{p}_2 - \vec{p}_1$$

$\lambda_1 = 0$

$B\vec{p}_1 = \lambda_1 \cdot \vec{p}_1$

$$e_1 = Be_0 = B(\vec{p}_2 - \vec{p}_1) = B\vec{p}_2 - B\vec{p}_1 =$$

$$= \lambda_2\vec{p}_2 - \cancel{\lambda_1}^{0}\cdot\vec{p}_1 = \frac{1}{4}\vec{p}_2$$

$\lambda_2 = \frac{1}{4}$

$B\vec{p}_2 = \lambda_2\vec{p}_2$

$$e_2 = Be_1 = B(\lambda_2\vec{p}_2) = \lambda_2 \cdot B\vec{p}_2 = \lambda_2^2\vec{p}_2$$

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$$e_k = \lambda_2^k\vec{p}_2 = \left(\frac{1}{4}\right)^k\vec{p}_2$$

We want to show that $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$

$$e_{k+1} = \lambda_2^{k+1}\vec{p}_2 = \lambda_2 \cdot \underbrace{\lambda_2^k\vec{p}_2} = \underset{\frac{1}{4}}{\lambda_2}\cdot e_k$$

$$\Rightarrow \|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$$

Def $\rho(B) = \{\max|\lambda|$ where λ is an eigenvalue of matrix $B\}$: <u>spectral radius</u>

Thm

If $\rho(B)<1$, then $x_n \to x$ for all initial guesses x_0.

Pf

Assume that matrix B has e'values $\lambda_1, .., \lambda_n$ with associated e'vectors $p_1, .., p_n$ that form a basis in $\mathbb{C}^n$ (this is true, for example, when B is symmetric).

For any x_0, $e_0 = x - x_0 = \alpha_1 p_1 + \ldots + \alpha_n p_n$

$$e_1 = Be_0 = B(\alpha_1 p_1 + \ldots + \alpha_n p_n) = \alpha_1 \underbrace{Bp_1}_{\lambda_1 p_1} + \ldots + \alpha_n \underbrace{Bp_n}_{\lambda_n p_n} =$$

$$= \alpha_1 \lambda_1 p_1 + \ldots + \alpha_n \lambda_n p_n$$

$$e_2 = Be_1 = B(\alpha_1 \lambda_1 p_1 + \ldots + \alpha_n \lambda_n p_n) =$$

$$= \alpha_1 \lambda_1^2 p_1 + \ldots + \alpha_n \lambda_n^2 p_n$$

$$\cdots\cdots\cdots$$

$$e_k = \alpha_1 \lambda_1^k p_1 + \ldots + \alpha_n \lambda_n^k p_n$$

$$\|e_k\| = \|\alpha_1 \lambda_1^k p_1 + \ldots + \alpha_n \lambda_n^k p_n\| \le$$

$$\le |\alpha_1| \cdot |\lambda_1|^k \|p_1\| + \ldots + |\alpha_n| \cdot |\lambda_n|^k \|p_n\|$$

Since $\rho(B) < 1 \Rightarrow |\lambda_1| < 1, |\lambda_2| < 1, \ldots, |\lambda_n| < 1$

$\Rightarrow |\lambda_1|^k \to 0, |\lambda_2|^k \to 0, \ldots, |\lambda_n|^k \to 0$ as $k \to \infty$

$\Rightarrow \|e_k\| \to 0$ as $k \to \infty$ $\Rightarrow$ $x_k \to x$ as $k \to \infty$

Note Proof also shows that

$$\|e_{k+1}\| \lesssim \rho(B) \cdot \|e_k\| \quad \text{as } k \to \infty$$

Recall

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ Then $B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow \rho(B_J) = \frac{1}{2}$

$\det(B_J - \lambda I) = 0$

$B_{GS} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \rho(B_{GS}) = \frac{1}{4}$

$\begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{1}{2}\right)^2$

Summary

1. $\|e_{k+1}\| \leq \|B\| \cdot \|e_k\|$: always
2. $\|e_{k+1}\| \lesssim \rho(B) \cdot \|e_k\|$: true if B is diagonalizable and $|\lambda_1| < 1, .., |\lambda_n| < 1$

Q What is the relation between $\|B\|$ and its spectral radius $\rho(B)$?

1. $\rho(B)$ is not a norm

Pf Take $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho(B) = 0$ but $B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2. $\rho(B) \leq \|B\|$