3/22/2010

Two-dimensional problems

$- \Delta \phi = f \quad$ for $(x,y) \in D \subset \mathbb{R}^2$: Poisson eq^n

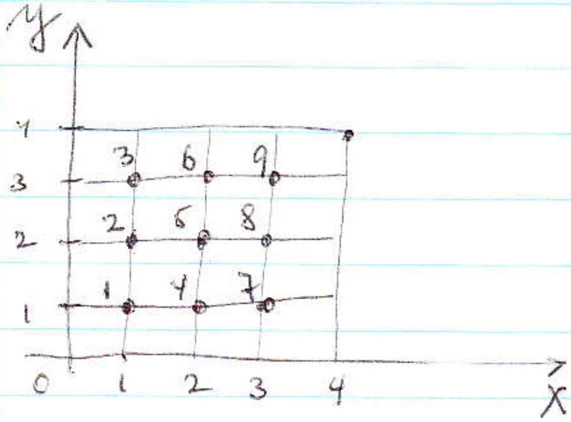
$\Delta \phi = \nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}$: Laplace operator

$\phi(x,y)$: unknown

$f(x,y)$: data - known

$\phi = g(x,y)$ for $(x,y) \in \partial D$: Dirichlet BC

Ex $\quad D = [0,1] \times [0,1]$



Uniform mesh : (ih, jh), $\quad i,j = 0,1,\dots, n+1$, $\quad h = \dfrac{1}{n+1}$

$\phi(ih, jh) \approx u_{ij}$

$f(ih, jh) = f_{ij}$

$$-\left(D_+^x D_-^x u_{ij} + D_+^y D_-^y u_{ij}\right) = f_{ij} \; : \; \text{5-point discrete Laplacian}$$

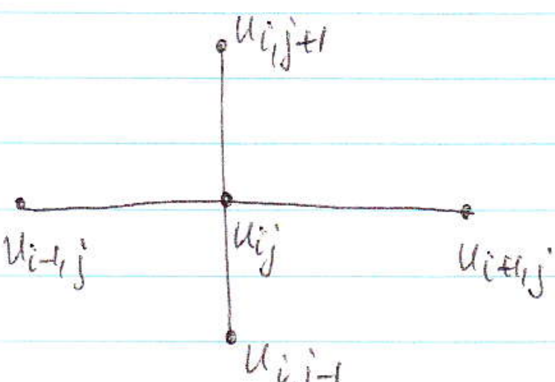
$$-\left(\frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2}\right) = $$

$$= f_{ij}$$

$i=1$
$j=2$

$$\frac{1}{h^2}\left(4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right) = f_{ij}$$

5 point stencil



Consider $\quad (i,j) = (1,1)$

$$\frac{1}{h^2}\left(4u_{11} - \underset{\substack{\downarrow \\ g_{01}}}{u_{01}} - u_{21} - \underset{\substack{\downarrow \\ g_{10}}}{u_{10}} - u_{12}\right) = f_{11}$$

$$\frac{1}{h^2}\left(\underset{\substack{\downarrow \\ \text{pt } 1}}{4u_{11}} - \underset{\substack{\downarrow \\ 4}}{u_{21}} - \underset{\substack{\downarrow \\ 2}}{u_{12}}\right) = f_{11} + \frac{1}{h^2}\left(g_{01} + g_{10}\right)$$

Lexicographic ordering

1	2	3	4	5	6	7	8	9
u_{11}	u_{12}	u_{13}	u_{21}	u_{22}	u_{23}	u_{31}	u_{32}	u_{33}
4	−1		−1					
−1	4	−1		−1				
	−1	4			−1			
−1			4	−1		−1		
	−1		−1	4	−1		−1	
		−1		−1	4			−1
			−1			4	−1	
				−1		−1	4	−1
					−1		−1	4

$Au = f$

$$A = \frac{1}{h^2}\begin{pmatrix} T & -I & O \\ -I & T & -I \\ O & -I & T \end{pmatrix}$$: block ~~three~~ tri diagonal
symmetric
positive definite

Note

$$
\begin{matrix}
4 & -1 & \boxed{0} & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & \boxed{0} & -1 & 0 & 0 & 0 & 0 \\
\boxed{0} & -1 & 4 & \boxed{0} & \boxed{0} & -1 & 0 & 0 & 0 \\
-1 & \boxed{0} & \boxed{0} & 4 & -1 & \boxed{0} & -1 & 0 & 0 \\
0 & -1 & \boxed{0} & -1 & 4 & -1 & \boxed{0} & -1 & 0 \\
0 & 0 & -1 & \boxed{0} & -1 & 4 & \boxed{0} & \boxed{0} & -1 \\
0 & 0 & 0 & -1 & \boxed{0} & \boxed{0} & 4 & -1 & \boxed{0} \\
0 & 0 & 0 & 0 & -1 & \boxed{0} & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & \boxed{0} & -1 & 4
\end{matrix}
$$

Def If $a_{ij} = 0$ for $|i-j| > p$, A is called a band matrix and $2p+1$ is bandwidth

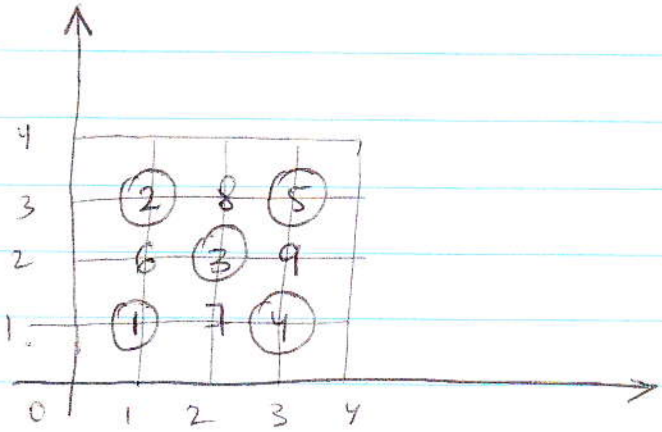
Ex $p = 3 \Rightarrow 2p+1 = 7$

Note As a result of LU factorization, fill-in occurs in the elements $\boxed{}$. LU factorization of a band matrix preserves the structure, but sparcity within the band will be lost. We have already seen this for a tridiagonal matrix.

Operation count

$$\#\text{ mult} \sim np^2 \ll \frac{n^3}{3}$$

Red-black ordering



$$
\begin{array}{ccccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
u_{11} & u_{13} & u_{22} & u_{31} & u_{33} & u_{12} & u_{21} & u_{23} & u_{32} \\
 & & & & & & & & \\
4 & & & & & -1 & -1 & & \\
 & 4 & & & & -1 & & -1 & \\
 & & 4 & & & -1 & -1 & -1 & -1 \\
 & & & 4 & & & -1 & & -1 \\
 & & & & 4 & & & -1 & -1 \\
\hline
-1 & -1 & -1 & & & 4 & & & \\
-1 & & -1 & -1 & & & 4 & & \\
 & -1 & -1 & & -1 & & & 4 & \\
 & & -1 & -1 & -1 & & & & 4
\end{array}
$$