3/24/2010

With red-black ordering equations that result from application of 5-point discrete Laplacian can be written as

B R B R
R B R B
B R B R
R B R B

$$\begin{pmatrix} D_R & H \\ K & D_B \end{pmatrix} \begin{pmatrix} u_R \\ u_B \end{pmatrix} = \begin{pmatrix} b_R \\ b_B \end{pmatrix}$$

where D_R, D_B are scalor diagonal matrices

Then

$$\boxed{\begin{aligned} u_R &= -D_R^{-1} H u_B + D_R^{-1} b_R \\ u_B &= -D_B^{-1} K u_R + D_B^{-1} b_B \end{aligned}}$$

$$D_R \cdot u_R + H u_B = b_R \qquad | \, D_R^{-1}$$

$$K u_R + D_B \cdot u_B = b_B \qquad | \cdot D_B^{-1}$$

$$\begin{cases} u_R + D_R^{-1} H u_B = D_R^{-1} b_R \\ u_B + D_B^{-1} K u_R = D_B^{-1} b_B \end{cases}$$

all red points can be calculated in parallel using black points. Then all black points can be calculated using red points

Polynomial approximation (Chapter 5)

Def A polynomial $p(x)$ of degree n has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \quad a_n \neq 0$$

The highest power, n in this case, is called the degree of the polynomial.

Ex $p(x) = 1 + x + x^2$ is a polynomial of degree 2.

Note

1. Any polynomial is a continuous function
2. But not all continuous functions are polynomials

   Ex $\sin x$

   $f(x) = \frac{1}{1+x^2}$ is continuous but it is not a polynomial

Thm (Weierstrass Thm)

Given a continuous function $f(x)$, $x \in [a,b]$. For any $\varepsilon > 0$, there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| \leq \varepsilon \quad \text{for all } x \in [a,b]$$

$$\max_{a \leq x \leq b} |f(x) - p(x)| \leq \varepsilon$$

application

$$\left| \int_a^b f(x)\,dx - \int_a^b p(x)\,dx \right| =$$

$$= \left| \int_a^b \big(f(x) - p(x) \big)\,dx \right| \leq \int_a^b |f(x) - p(x)|\,dx \leq$$

$$\leq \int_a^b \varepsilon\,dx = \varepsilon(b-a)$$

Taylor theorem

Let $f(x)$ be defined on $[a,b]$ and suppose $f^{(n+1)}(x)$ is continuous for all $x \in [a,b]$. Then if $x_0, x \in [a,b]$, then there exists $\xi = \xi(x)$ between x_0 and x such that

$$f(x) = p_n(x) + r(x)$$

where

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

n^{th} degree Taylor polynomial at pt $x = x_0$

$$r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad : \text{remainder or error}$$

Ex

$$f(x) = \frac{1}{1+25x^2}$$

$x_0 = 0$

$p_0 = 1$

$p_2 = 1 - 25x^2$

$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$

$z = -25x^2$

$p_4 = 1 - 25x^2 + 625x^4$

$p_6 = 1 - 25x^2 + 625x^4 - 15625x^6$

Note $\int_{-1}^{1} \underbrace{f(x)}_{= \frac{1}{1+25x^2}} dx$ is poorly approximated by $\int_{-1}^{1} p_n(x)\,dx$

$$f(x) = \frac{1}{1+25x^2}$$

$$p_{2n}(x) = \sum_{k=0}^{n} (-25x^2)^k = 1 - 25x^2 + 625x^4$$

$$- 15625x^6 + \dots + (-25x^2)^n$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \qquad |z| < 1$$

$$z = -25x^2 \Rightarrow |25x^2| < 1 \Rightarrow x^2 < \frac{1}{25}$$

$$\Rightarrow |x| < \frac{1}{5} = 0.2$$

Note

1. If $|x| < 0.2$, then $\lim_{n\to\infty} p_{2n} = f(x)$
2. If $|x| \not< 0.2$, then $\lim_{n\to\infty} p_{2n}(x)$ doesn't exist

Polynomial interpolation

Let f be a continuous function and $x_0, x_1, \dots, x_n$: distinct points

Questions

1. Does there exist a unique polynomial p