3/26/2010

Polynomial interpolation

We consider continuous function f and assume we have

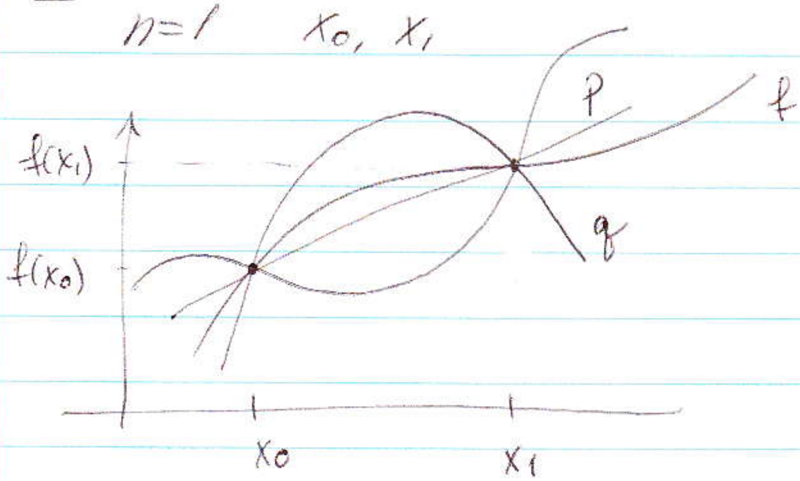
$x_0, x_1, \dots, x_n$: distinct points

Questions

1. Does there exist a unique polynomial p of least degree which interpolates function f at given points, i.e. such that $f(x_i) = p(x_i)$, $i = 0, 1, \dots, n$?

Ex

$n=1$ x_0, x_1



$\deg p = 1$
$\deg q = 2$

Thm (uniqueness)

If $x_0, x_1, \dots, x_n$ are $n+1$ distinct points and p, q are polynomials of degree $\le n$ such that $p(x_i) = q(x_i)$ for $i = 0, \dots, n$, then $p(x) = q(x)$ for all x.

Pf

Use fundamental theorem of algebra ...
(n^{th} degree polynomial has exactly n roots)
Here consider $h(x) = p(x) - q(x)$

$h(x_i) = 0$, $i = 0, 1, .., n$: $n+1$ points

2. What is the best way to evaluate $p(x)$ at $x \neq x_i$?

3. How large is the error $|f(x) - p(x)|$ at $x \neq x_i$?

Def Let $\mathcal{P}_n$ the set of polynomials of degree $\leq n$

$$\mathcal{P}_n = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k, \ k \leq n, \ a_i \in \mathbb{R} \}$$

Note

$\mathcal{P}_n$ is a vector space over $\mathbb{R}$
since if $p, q \in \mathcal{P}_n \Rightarrow p+q \in \mathcal{P}_n$
$\alpha p \in \mathcal{P}_n$, $\alpha \in \mathbb{R}$

$a + bx$

$\dim \mathcal{P}_n = n+1$

The standard basis for $\mathcal{P}_n$ is $\{1, x, x^2, .., x^n\}$

$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$: in standard basis

Def

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \quad k = 0, 1, \dots, n:$$

Lagrange polynomials

Ex

$n = 2, \quad x_0 = 1, \quad x_1 = 2, \quad x_2 = 3$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2-5x+6}{2}$$

$$= \frac{1}{2}x^2 - \frac{5}{2}x + 3$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2-4x+3}{-1}$$

$$= -x^2 + 4x - 3$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2-3x+2}{2}$$

$$= \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

Note

1. $\deg l_k(x) = n$

2. $l_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$

Given f: continuous, $x_0, x_1, \dots, x_n$: $n+1$ distinct points, define

Interpolating polynomial in Lagrange form

$$p_n(x) = \sum_{k=0}^{n} f(x_k) \cdot l_k(x) =$$

$$= f(x_0) l_0(x) + f(x_1) l_1(x) + \dots + f(x_n) l_n(x)$$

Note

1. $\deg p_n \le n$

2. $p_n(x_i) = \sum_{k=0}^{n} f(x_k) \cdot l_k(x_i) = f(x_i), \quad i = 0, 1, \dots, n$

Thus, $p_n(x)$ interpolates $f(x)$ at $x_0, x_1, \dots, x_n$ and has degree $\le n$.

Ex $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $n = 2$

$$p_2(x) = f(x_0) l_0(x) + f(x_1) l_1(x) + f(x_2) l_2(x)$$

$$= 1 \cdot \left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right) + \frac{1}{2}\left(-x^2 + 4x - 3\right) +$$

$$+ \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right) = \frac{1}{6}x^2 - x + \frac{11}{6}$$

$p_2(1) = \frac{1}{6} - 1 + \frac{11}{6} = 1$ ✓

$p_2(2) = \frac{1}{6} \cdot 2^2 - 2 + \frac{11}{6} = \frac{4}{6} - \overset{12}{2} + \frac{11}{6} = \frac{4 - 12 + 11}{6} = \frac{3}{6} = \frac{1}{2}$ ✓

$p_3(3) = \frac{1}{3}$ ✓

Note

Lagrange form of interpolating polynomial $p_n(x)$ is good because it shows that interpolating polynomial exists, but it has computational disadvantages:

1. it is expensive to evaluate $p_n(x)$ at $x \neq x_i$

2. if we want to include an additional point x_{n+1}, we have to recompute all Lagrangian polynomials $l_k(x)$.

Def

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$

$$\dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

This is an interpolating polynomial in Newton's form

Ex $n=1$, x_0, x_1

$$p_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) =$$

$$= f(x_0)\cdot \frac{x-x_1}{x_0-x_1} + f(x_1)\cdot \frac{x-x_0}{x_1-x_0} \quad : \text{ Lagrange form}$$

$$= f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) \quad : \text{ Newton's form}$$

Note

If $n>1$, we need to have an algorithm of computing coefficients $a_0, a_1, \ldots, a_n$ in Newton's form of interpolating polynomial p_n.

Claim

There exists a number a_n such that

$$p_n(x) = p_{n-1}(x) + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

Pf

$$p_n(x_i) = p_{n-1}(x_i) + a_n(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{n-1})$$

If $0 \le i \le n-1 \Rightarrow p_n(x_i) = p_{n-1}(x_i) + a_n \cdot 0$

For $i=n$:

$$f(x_n) = p_n(x_n) = p_{n-1}(x_n) + a_n\underbrace{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}_{\neq 0}$$

$$f(x_n) = p_{n-1}(x_n) + a_n (x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})$$

$$\Rightarrow \quad a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

$\leftarrow$ gives equation to find a_n