3/29/2010

Notation

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots$$

$$\ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$: interpolating polynomial in Newton's form

$a_0 = f[x_0]$
$a_1 = f[x_0, x_1]$
$a_2 = f[x_0, x_1, x_2]$
$- \; - \quad - \; -$
$a_n = f[x_0, x_1, x_2, \ldots, x_n]$

Then

Newton's form of interpol. polynomial

$$p_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$

$$+ \ldots + f[x_0, x_1, \ldots, x_n](x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

<u>Note</u> It is easy to include an additional point x_{n+1}. The previous work will not be wasted, i.e. previous terms will not be changed.

<u>Claim</u>

divided differences

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

Proof

$p_{n-1}(x)$: interpolates f at $x_0, x_1, .., x_{n-1}$: $\deg p_{n-1} \leq n-1$

$q_{n-1}(x)$: interpolates f at $x_1, x_2, .., x_n$: $\deg q_{n-1} \leq n-1$

Define

$$s(x) = \frac{x - x_0}{x_n - x_0} q_{n-1}(x) + \frac{x_n - x}{x_n - x_0} p_{n-1}(x)$$

$\deg s \leq n$

$s(x_0) = p_{n-1}(x_0) = f(x_0)$

$s(x_n) = q_{n-1}(x_n) = f(x_n)$

For $1 \leq i \leq n-1$

$$s(x_i) = \frac{x_i - x_0}{x_n - x_0} \underbrace{q_{n-1}(x_i)}_{f(x_i)} + \frac{x_n - x_i}{x_n - x_0} \underbrace{p_{n-1}(x_i)}_{f(x_i)} =$$

$$= f(x_i) \left(\frac{\cancel{x_i} - x_0}{x_n - x_0} + \frac{x_n - \cancel{x_i}}{x_n - x_0} \right) = f(x_i) \quad \checkmark$$

$\Rightarrow s(x)$ interpolates f at $x_0, x_1, .., x_n$

$\Rightarrow s(x) = p_n(x)$ by uniqueness

Evaluate coefficients of x^n:

$$\frac{1}{x_n - x_0} f[x_1, .., x_n] - \frac{1}{x_n - x_0} f[x_0, .., x_{n-1}] = f[x_0, x_1, ..., x_n]$$

$\blacksquare$

Divided difference table

$$
\begin{tikzcd}[row sep=small]
x_0 & f[x_0] \arrow[dr] & & \\
 & & f[x_0, x_1] \arrow[dr] & \\
x_1 & f[x_1] \arrow[ur] \arrow[dr] & & f[x_0, x_1, x_2] \\
 & & f[x_1, x_2] \arrow[ur] & \\
x_2 & f[x_2] \arrow[ur] & &
\end{tikzcd}
$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Then

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

i.e. the circled numbers are the coefficients of the interpolating polynomial in Newton's form

If we want to include an extra pt x_3, then

$$\begin{array}{lllll}
x_0 & f[x_0] & & & \\
 & & f[x_0,x_1] & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & \\
 & & f[x_1,x_2] & & f[x_0,x_1,x_2,x_3] \\
x_2 & f[x_2] & & f[x_1,x_2,x_3] & \\
 & & f[x_2,x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

Ex $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

x_i	$f[.]$	$f[.,.]$	$f[.,.,.]$
1	1		
		$\frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}$	
2	$\frac{1}{2}$		$\frac{-\frac{1}{6}-\left(-\frac{1}{2}\right)}{3-1} = \frac{-\frac{1}{6}+\frac{1}{2}}{2} = \frac{-1+3}{6\cdot 2} = \frac{1}{6}$
		$\frac{\frac{1}{3}-\frac{1}{2}}{3-2} = -\frac{1}{6}$	
3	$\frac{1}{3}$		

$$p_2(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2)$$

Evaluation of $p_n(x)$

$$p_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$: Newton's form, 3 mult

$$p_2(x) = a_0 + (x-x_0)\left(a_1 + a_2 \cdot (x-x_1)\right)$$: nested form, 2 mult

General case

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$
$$\dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$p_n(x) = a_0 + (x-x_0)\Big(a_1 + (x-x_1)\Big(a_2 + \dots$$
$$\dots + a_n(x-x_{n-1})\dots\Big)$$

Operation count

\# mult $= \dfrac{n(n+1)}{2}$: Newton's form

n : nested form