

3/29/2010

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Notation

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$

$\dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$: interpolating polynomial in Newton's form

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1]$$

$$a_2 = f[x_0, x_1, x_2]$$

$$\dots$$
$$a_n = f[x_0, x_1, x_2, \dots, x_n]$$

Then

Newton's form

of interpol.

polynomial

$$p_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$

$$+ \dots + f[x_0, x_1, \dots, x_n](x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Note It is easy to include an additional point x_{n+1} . The previous work will not be wasted, i.e. previous terms will not be changed.

Claim

divided differences

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$$

Proof

$p_{n-1}(x)$: interpolates f at x_0, x_1, \dots, x_{n-1} : $\deg p_{n-1} \leq n-1$

$q_{n-1}(x)$: interpolates f at x_1, x_2, \dots, x_n : $\deg q_{n-1} \leq n-1$

Define

$$s(x) = \frac{x-x_0}{x_n-x_0} q_{n-1}(x) + \frac{x_n-x}{x_n-x_0} p_{n-1}(x)$$

$\deg s \leq n$

$$s(x_0) = p_{n-1}(x_0) = f(x_0)$$

$$s(x_n) = q_{n-1}(x_n) = f(x_n)$$

For $1 \leq i \leq n-1$

$$s(x_i) = \frac{x_i-x_0}{x_n-x_0} \underbrace{q_{n-1}(x_i)}_{f(x_i)} + \frac{x_n-x_i}{x_n-x_0} \underbrace{p_{n-1}(x_i)}_{f(x_i)} =$$

$$= f(x_i) \left(\frac{x_i-x_0}{x_n-x_0} + \frac{x_n-x_i}{x_n-x_0} \right) = f(x_i) \quad \checkmark$$

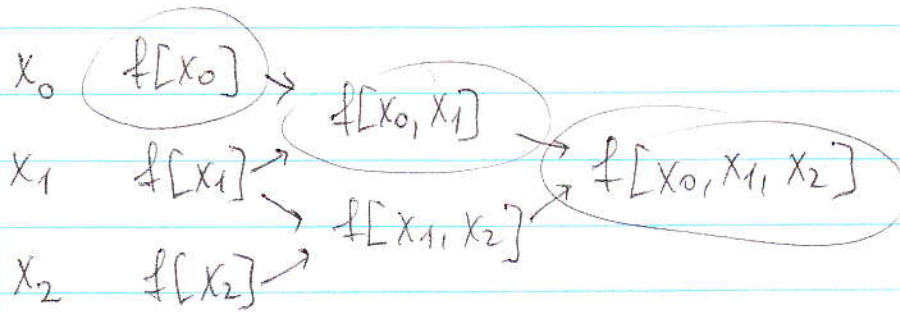
$\Rightarrow s(x)$ interpolates f at x_0, x_1, \dots, x_n

$\Rightarrow s(x) = p_n(x)$ by uniqueness

evaluate coefficients of x^n :

$$\frac{1}{x_n-x_0} f[x_1, \dots, x_n] - \frac{1}{x_n-x_0} f[x_0, \dots, x_{n-1}] = f[x_0, x_1, \dots, x_n]$$

Divided difference table



$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

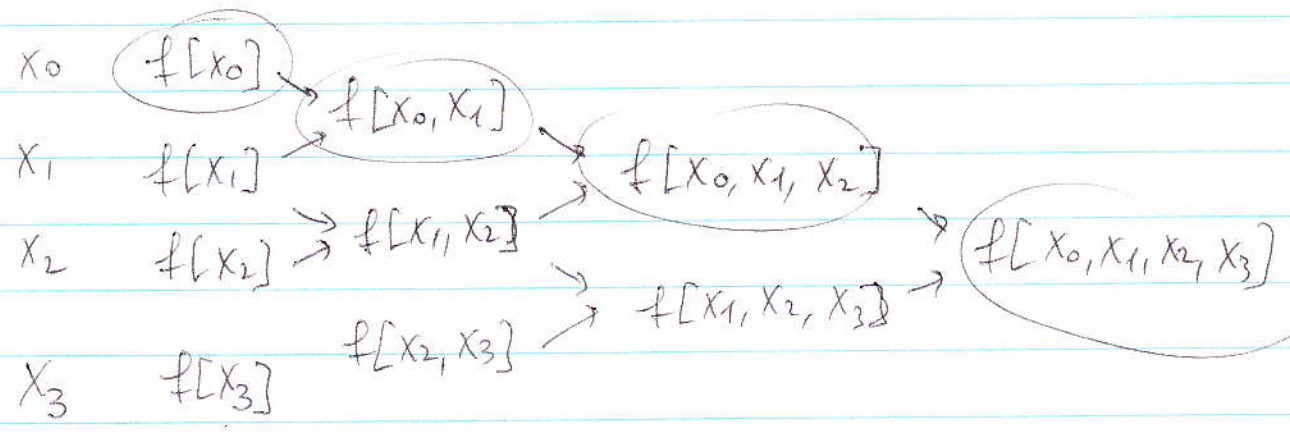
$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Then

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

i.e. the circled numbers are the coefficients of the interpolating polynomial in Newton's form

If we want to include an extra pt x_3 , then



$$p_3(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

Ex $f(x) = \frac{1}{x}, x_0=1, x_1=2, x_2=3$

x_i	$f[.]$	$f[.,.]$	$f[.,.,.]$
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1	1		
2	$\frac{1}{2}$	$\frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}$	$\frac{-\frac{1}{6}-(-\frac{1}{2})}{3-1} = \frac{-\frac{1}{6}+\frac{1}{2}}{2} = \frac{-1+3}{6 \cdot 2} = \frac{1}{6}$
3	$\frac{1}{3}$	$\frac{\frac{1}{3}-\frac{1}{2}}{3-2} = -\frac{1}{6}$	

$$p_2(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2)$$

Evaluation of $p_n(x)$

$$p_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) : \text{Newton's form}$$

3 mult

$$p_2(x) = a_0 + (x-x_0) \left(a_1 + a_2 \cdot (x-x_1) \right) : \text{nested form}$$

2 mult

General case

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$

$$\dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$p_n(x) = a_0 + (x-x_0) \left(a_1 + (x-x_1) \left(a_2 + \dots \right. \right. \\ \left. \left. \dots + a_n(x-x_{n-1}) \dots \right) \right)$$

Operation count

$$\# \text{ mult} = \frac{n(n+1)}{2} : \text{Newton's form}$$

$$n : \text{nested form}$$