2. $\rho(B) \le \|B\|$ for any matrix norm

Pf

Suppose λ is an e'value of B with e'vector p, i.e. $Bp = \lambda p$, $p \neq 0$

Then $|\lambda| \cdot \|pp^T\| = \|\lambda pp^T\| = \|Bp \cdot p^T\| = \|B \cdot pp^T\| \le$

$\le \|B\| \cdot \|pp^T\|$

$\Rightarrow$ since $p \neq 0 \Rightarrow$ we can divide by $\|pp^T\|$

$\Rightarrow$ $|\lambda| \le \|B\|$ true for any e'value of B

$\Rightarrow$ $\underbrace{|\lambda|_{max}}_{\rho(B)} \le \|B\| \Rightarrow \rho(B) \le \|B\|$

3. $\rho(B) = \lim_{k \to \infty} \|B^k\|^{1/k}$

This result is useful since $e_k = B^k e_0$

Ex $B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow \rho(B_J) = \frac{1}{2} \le \|B_J\|_\infty = \frac{1}{2}$ $\quad \lambda_{1,2} = \pm \frac{1}{2}$

$B_{GS} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \lambda = 0$ and $\lambda = \frac{1}{4} \Rightarrow \rho(B_{GS}) = \frac{1}{4} \le \underbrace{\|B_{GS}\|_\infty}_{\frac{1}{2}}$

Relaxation

$Ax = b, \quad A = L + D + U$

$$Ax = b \Rightarrow (L + D + U)x = b$$

$$(L + D)x = -Ux + b$$

Gauss-Seidel method:

$$(L+D)\, x_{k+1} = -U x_k + b$$

One of the forms of Gauss-Seidel method is

used to update components

$$D x_{k+1} = D x_k - \left(L x_{k+1} + (D+U) x_k - b \right)$$

Let ω be an <u>acceleration parameter</u>

$$D x_{k+1} = D x_k - \omega \left(L x_{k+1} + (D+U) x_k - b \right)$$

$$(\omega L + D)\, x_{k+1} = D x_k - \omega (D+U) x_k + \omega b$$

$$(\omega L + D)\, x_{k+1} = \left(D - \omega (D+U) \right) x_k + \omega b$$

$$(\omega L + D)\, x_{k+1} = \left((1-\omega) D - \omega U \right) x_k + \omega b$$

$$B_\omega = (\omega L + D)^{-1} \left((1-\omega) D - \omega U \right)$$

<u>Note</u> when $\omega = 1$ we recover Gauss-Seidel

Components

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$a_{11} x_1^{(k+1)} = a_{11} x_1^{(k)} - \omega \left(a_{11} x_1^{(k)} + a_{12} x_2^{(k)} + a_{13} x_3^{(k)} - b_1 \right)$$

$$a_{22} x_2^{(k+1)} = a_{22} x_2^{(k)} - \omega \left(a_{21} x_1^{(k+1)} + a_{22} x_2^{(k)} + a_{23} x_3^{(k)} - b_2 \right)$$

$$a_{33} x_3^{(k+1)} = a_{33} x_3^{(k)} - \omega \left(a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k)} - b_3 \right)$$

Note $\quad 1 < \omega < 2$

When $\omega > 1$, the method is called successive overrelaxition (SOR). It is used to accelerate convergence for those systems that converge using Gauss-Seidel method

$0 < \omega < 1$: under-relaxation methods.
Used to obtain convergence for those systems for which Gauss-Seidel doesn't converge.

Ex
$$2x_1 - x_2 = 1$$
$$-x_1 + 2x_2 = 1$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$2 x_1^{(k+1)} = 2 x_1^{(k)} - \omega \left(2x_1^{(k)} - x_2^{(k)} - 1 \right)$$

$$2 x_2^{(k+1)} = 2 x_2^{(k)} - \omega \left(-x_1^{(k+1)} + 2 x_2^{(k)} - 1 \right)$$

$$\begin{pmatrix} 2 & 0 \\ -\omega & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{k+1} = \begin{pmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_k + \omega \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$B_\omega = \begin{pmatrix} 2 & 0 \\ -\omega & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{pmatrix} = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4}+1-\omega \end{pmatrix}$$

<u>Note</u> : $\omega = 1 \Rightarrow \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} = B_{GS} \Rightarrow \rho(B_{GS}) = \frac{1}{4}$

(Gauss-Seidel)

$$\omega = \frac{4}{2+\sqrt{3}} \sim 1.0718 \Rightarrow B_\omega = \begin{pmatrix} -0.0718 & 0.5359 \\ -0.0385 & 0.2154 \end{pmatrix}$$

$$\Rightarrow \rho(B_\omega) = 0.07$$

GS, $\lVert e_k \rVert_\infty$	k	x_1^k	x_2^k	$\lVert e_k \rVert_\infty$	$\lVert e_k \rVert_\infty / \lVert e_{k-1} \rVert_\infty$
1	0	0.0000	0.0000	1.0000	—
$\frac{1}{2} = 0.5$	1	0.5359	0.8231	0.4641	0.4641
$\frac{1}{8} = 0.125$	2	0.9385	0.9798	0.0615	0.1325
$\frac{1}{32} = 0.03125$	3	0.9936	0.9980	0.0064	0.1047
$\frac{1}{128} = 0.00731$	4	0.9994	0.9998	0.0006	0.0944