3/5/2010

Thm

If $\rho(B_\omega) < 1$, then $0 < \omega < 2$.

Pf We will prove this result for our example of 2×2 matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad B_\omega = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4} + 1 - \omega \end{pmatrix}$$

We will need to use a result about matrices: $\det A = \lambda_1 \lambda_2 \dots \lambda_n$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are e'values of A.

Let λ_1, λ_2 be e'values of B_ω.

If $\rho(B_\omega) < 1 \Rightarrow |\lambda_1| < 1, \ |\lambda_2| < 1$

$$\Rightarrow \underbrace{|\lambda_1 \cdot \lambda_2|}_{= |\det B_\omega|} < 1 \Rightarrow |\det B_\omega| < 1$$

$$\det B_\omega = (1-\omega)\left(\frac{\omega^2}{4} + 1 - \omega\right) - \frac{\omega^2}{4}(1-\omega) =$$

$$= (1-\omega)\left(\cancel{\frac{\omega^2}{4}} + 1 - \omega - \cancel{\frac{\omega^2}{4}}\right) = (1-\omega)^2$$

$$\Rightarrow |\det B_\omega| < 1 \Rightarrow (1-\omega)^2 < 1$$

or $|1-\omega| < 1 \Rightarrow -1 < 1-\omega < 1$

or $|\omega - 1| < 1 \Rightarrow -1 < \omega - 1 < 1 \quad / +1$

$$\Rightarrow \boxed{0 < \omega < 2}$$

QE

Thm

Let A be block tridiagonal, symmetric, positive definite. Define

$$\omega_* = \frac{2}{1+\sqrt{1-\rho(B_J)^2}} \quad : \text{optimal SOR parameter}$$

Then

$$\rho(B_{\omega_*}) = \min_{0<\omega<2} \rho(B_\omega) = \omega_* - 1 < \rho(B_{GS}) <$$

$$< \rho(B_J) < 1$$

Ex $\rho(B_J) = \frac{1}{2} \Rightarrow \omega_* = \frac{4}{2+\sqrt{3}} \left(= \frac{2}{1+\sqrt{1-\left(\frac{1}{2}\right)^2}} \right)$

Recall

(sdd)

A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad i=1,\dots,n$$

Ex $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$: sdd

Thm

If A is strictly diagonally dominant, then

1. $\|B_J\|_\infty < 1$

2. $\|B_{GS}\|_\infty < 1$

Pf

1. $A = L + D + U$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}| \stackrel{def}{=}$$

$$= l_i + u_i$$

This is just notation

Then $\dfrac{l_i + u_i}{|a_{ii}|} < 1$ for $i = 1, \dots, n$

However,

$$B_J = -D^{-1}(L+U) \Rightarrow$$

$$\Rightarrow \|B_J\|_\infty = \max_i \left\{ \frac{l_i + u_i}{|a_{ii}|} \right\} < 1 \quad \text{or}$$

$$D^{-1} = \operatorname{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{nn}} \right)$$

2. HW Consider 2x2 case

Analysis of SOR iteration matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_\omega = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4} + 1-\omega \end{pmatrix}$$

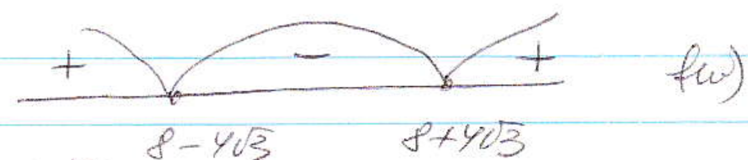
Eigenvalues of B_ω (solutions of $\det(B_\omega - \lambda I) = 0$) are:

$$\lambda_{1,2} = 1-\omega + \frac{1}{8}\omega^2 \pm \frac{1}{8}\omega\sqrt{16 - 16\omega + \omega^2}$$

$$f(\omega) = \omega^2 - 16\omega + 16 = 0$$

$$\frac{16 \pm \sqrt{16^2 - 4 \cdot 16}}{2}$$

$$16^2 - 4 \cdot 16 = 16(16-4) = 16 \cdot 12 > 0$$



$$\omega_{1,2} = 8 \pm 4\sqrt{3}$$