Nonlinear Systems (2x2)

$f(x,y)=0$

$g(x,y)=0$

$$\cancel{f(x_{n+1},y_{n+1})}^{\,0} \overset{\text{Taylor}}{=} f(x_n,y_n) + f_x(x_n,y_n)(x_{n+1}-x_n) + f_y(x_n,y_n)(y_{n+1}-y_n) + \cancel{\dots}^{\,0}$$

expand f around (x_n,y_n)

$$\cancel{g(x_{n+1},y_{n+1})}^{\,0} = g(x_n,y_n) + g_x(x_n,y_n)(x_{n+1}-x_n) + g_y(x_n,y_n)(y_{n+1}-y_n) + \cancel{\dots}^{\,0}$$

$$f_x(x_n,y_n)(x_{n+1}-x_n) + f_y(x_n,y_n)(y_{n+1}-y_n) = -f(x_n,y_n)$$

$$g_x(x_n,y_n)(x_{n+1}-x_n) + g_y(x_n,y_n)(y_{n+1}-y_n) = -g(x_n,y_n)$$

$$(1) \quad \overbrace{\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_n,y_n)}}^{J} \cdot \underbrace{\begin{pmatrix} x_{n+1}-x_n \\ y_{n+1}-y_n \end{pmatrix}}_{\equiv\, p} = -\overbrace{\begin{pmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{pmatrix}}^{=\,v}$$

Jacobian matrix

given $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Given $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ eqn (1) is a Newton's method for system of 2 equations.

We do not invert matrix to find $\begin{pmatrix} x_{n+1}-x_n \\ y_{n+1}-y_n \end{pmatrix}$ but solve the system using direct or iterative methods

$$J \cdot p = v \qquad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Compute vector p at every iteration. Then update your solution

known from previous iteration

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_n = \begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix}$$

computed by solving linear system

$$\Rightarrow \quad x_{n+1} = x_n + p_1$$

$$y_{n+1} = y_n + p_2$$

<u>Note</u> 1. If there is only one eq^{n} then this method reduces to Newton's method for scalar eq$^{\underline{ns}}$.

2. The method is sensitive to the initial guess