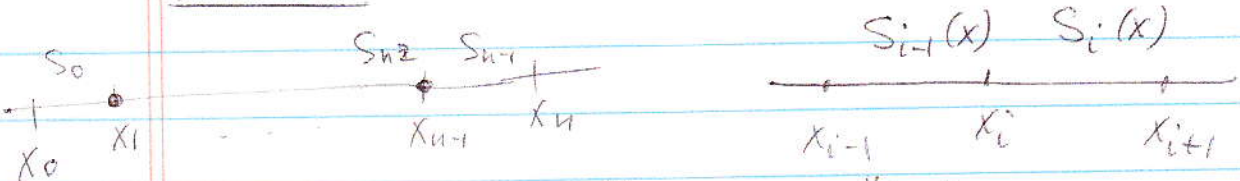
4/12/2010

Given $f(x)$, $x_0, x_1, \ldots, x_n$: distinct points.
How to construct spline $S(x)$?
cubic

<u>Ex</u> $x_i = ih$, $h = \frac{1}{n}$, $x_0 = 0$, $x_n = 1$

<u>Step 1</u> (2nd derivative condition)



We enforce: $S''_{i-1}(x_i) = S''_i(x_i)$

Since $S_i(x)$ is cubic polynomial, $S''_i(x)$ is a linear function in x

(*) Lagrange interpol. polynomial of degree 1

$$\boxed{S''_i(x) = a_i \frac{x_{i+1} - x}{h} + a_{i+1} \frac{x - x_i}{h}, \quad i = 0, \ldots, n-1}$$

Then

$$S''_i(x_i) = a_i \frac{x_{i+1} - x_i}{h} + a_{i+1} \frac{\overbrace{x_i - x_i}^{\to 0}}{h} = a_i$$

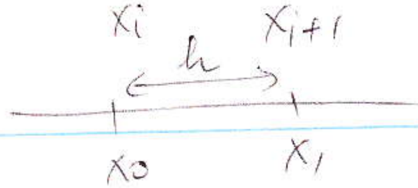
$$S''_i(x_{i+1}) = a_i \frac{\overbrace{x_{i+1} - x_{i+1}}^{\to 0}}{h} + a_{i+1} \frac{x_{i+1} - x_i}{h} = a_{i+1}$$

$i \to i-1$ $\quad S''_{i-1}(x_i) = a_i$

$$\Rightarrow S''_i(x_i) = a_i = S''_{i-1}(x_i)$$

Thus, $S''_i(x)$ is continuous at interior points $x_1, \ldots, x_{n-1}$.

Justification for (*)



$n=1 \qquad x_0, \quad x_1$

$$l_0(x) = \frac{x-x_1}{x_0-x_1} \qquad l_1(x) = \frac{x-x_0}{x_1-x_0}$$

$$p_1 = f = \underbrace{f(x_0)}\, l_0(x) + f(x_1)\, l_1(x)$$

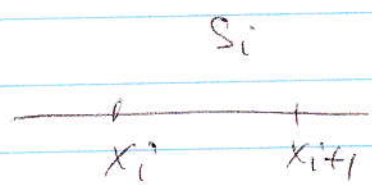
$$a_i\, l_0(x) + a_{i+1}\, l_1(x) = a_i \frac{x-x_{i+1}}{\underbrace{x_i - x_{i+1}}_{-h}} + a_{i+1} \frac{x-x_i}{\underbrace{x_{i+1}-x_i}_{h}}$$

Step 2 (function values)

Integrate twice

$$S_i(x) = \frac{a_i (x_{i+1}-x)^3}{6h} + \frac{a_{i+1}(x-x_i)^3}{6h} +$$

$$+ \underbrace{b_i (x_{i+1}-x) + c_i (x-x_i)}_{\text{for convenience}}$$



$$\left.\begin{aligned} S_i(x_i) &= \frac{a_i h^2}{6} + b_i h = f_i \\ S_i(x_{i+1}) &= \frac{a_{i+1} h^2}{6} + c_i h = f_{i+1} \end{aligned}\right\}$$

$$\Rightarrow \quad b_i h = f_i - \frac{a_i h^2}{6}$$

$$c_i h = f_{i+1} - \frac{a_{i+1} h^2}{6}$$

Substitute:

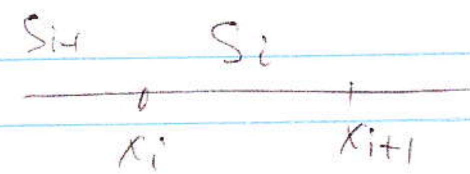
$$S_i(x) = \frac{a_i (x_{i+1}-x)^3}{6h} + \frac{a_{i+1}(x-x_i)^3}{6h} + \left(\frac{f_i}{h} - \frac{a_i h}{6}\right)(x_{i+1}-x)$$

$$+\left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)(x-x_i)$$

<u>Step 3</u> (1st derivative condition)

$$S_i'(x) = -\frac{a_i (x_{i+1}-x)^2}{2h} + \frac{a_{i+1}(x-x_i)^2}{2h} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) +$$

$$+\left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$



$$S_i'(x_i) = -\frac{a_i h}{2} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$

$$S_i'(x_{i+1}) = \frac{a_{i+1} h}{2} - \left(\frac{f_i}{h} - \frac{a_i h}{6}\right) + \left(\frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}\right)$$

$i \to i-1$

$$S_{i-1}'(x_i) = \frac{a_i h}{2} - \left(\frac{f_{i-1}}{h} - \frac{a_{i-1} h}{6}\right) + \left(\frac{f_i}{h} - \frac{a_i h}{6}\right)$$

We require, $S_{i-1}'(x_i) = S_i'(x_i) \Rightarrow$

$$-\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6} = \frac{a_i h}{2} -$$

$$-\frac{f_{i-1}}{h} + \frac{a_{i-1} h}{6} + \frac{f_i}{h} - \frac{a_i h}{6}$$

Simplify to get

$$a_{i-1}\frac{h}{6} + a_i\left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right) + a_{i+1}\frac{h}{6} =$$

$$= \left(f_{i-1} - 2f_i + f_{i+1}\right)/h \quad \Big| \cdot \frac{6}{h}$$

$$(**) \qquad a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right)$$

This holds for $i = 1, 2, \ldots, n-1$

Step 4 (boundary conditions)

$S_0''(x_0) = 0 \Rightarrow a_0 = 0$

$S_{n-1}''(x_n) = 0 \Rightarrow a_n = 0$

$$\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

Coefficient Matrix is symmetric, tridiagonal, strictly diagonally dominant, positive definite

$\Rightarrow$ The above system has a solution

Recall

$A=(a_{ij})$ is strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad i=1,2,\dots,n$$

(sdd)

Thm Strictly diagonally dominant matrix is invertible.

In practice, coefficients $a_1, a_2, \dots, a_{n-1}$ are computed by using Gaussian elimination of a tridiagonal matrix

Note Clamped BCs: $S_0'(x_0) = f'(x_0)$

$$S_{n-1}'(x_n) = f'(x_n)$$

Coefficients a_0 and a_n are not zero anymore.

$i=0$

$$S_0'(x) = -\frac{a_0(x_1-x)^2}{2h} + \frac{a_1(x-x_0)^2}{2h} - \left(\frac{f_0}{h} - \frac{a_0 h}{6}\right) + \left(\frac{f_1}{h} - \frac{a_1 h}{6}\right)$$

$$S_0'(x_0) = -\frac{a_0}{2} - \left(\frac{f_0}{h} - \frac{a_0 h}{6}\right) + \left(\frac{f_1}{h} - \frac{a_1 h}{6}\right) = f_0'$$

gives extra eq^{n} for a_0 and a_1

Similarly, condition $S_{n-1}'(x_n) = f_n'$ gives another eq^{n} for a_{n-1} and a_n.

eq^{ns} (**) together with two previous conditions will form a system of $n+1$ eq^{ns} for unknowns $a_0, a_1, \ldots, a_n$