HW # 9

4

$x_0 \quad f(x_0) \quad f'(x_0) \quad f''(x_0)$

$x_1 \quad f(x_1) \quad f'(x_1)$

	$z_0 = x_0$	$f(x_0)$				
			$f'(x_0)$			
$x_0+\varepsilon$	$z_1 = x_0$	$f(x_0)$		$\frac{1}{2!} f''(x_0)$		
			$f'(x_0)$		$f[z_0,z_1,z_2,z_3]$	
$x_0+2\varepsilon$	$z_2 = x_0$	$f(x_0)$		$\dfrac{f[x_0,x_1]-f'(x_0)}{z_3-z_1}$		$\boxed{\ }$
			$f[x_0,x_1]$		$f[z_1,z_2,z_3,z_4]$	
	$z_3 = x_1$	$f(x_1)$		$\dfrac{f'(x_1)-f[x_0,x_1]}{z_4-z_2}$		
			$f'(x_1)$			
	$z_4 = x_1$	$f(x_1)$				

$$p(x) = f(x_0) + f'(x_0)(x-z_0) + \frac{1}{2!} f''(x_0)(x-z_0)(x-z_1) +$$

$$+ (x-z_0)(x-z_1)(x-z_2) + \boxed{\ }\,(x-z_0)(x-z_1)(x-z_2) *$$

$$*(x-z_3)$$

Thm

Let $f(x)$ be defined on $[a,b]$,
$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and let $S(x)$ be
a cubic spline interpolant of f with natural
or clamped boundary conditions

1. $$|f(x) - S(x)| \le \frac{5}{384} \max_{a \le x \le b} |f^{(4)}(x)| \cdot h^4$$

where $h = \max_i |x_{i+1} - x_i|$

2. $$\int_a^b [S''(x)]^2\,dx \le \int_a^b [f''(x)]^2\,dx$$

The first condition says that spline
interpolation is 4^{th} order accurate

Recall

$$\kappa(x) = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}} \approx |f''(x)|$$

$\int_a^b [f''(x)]^2\,dx$ is a crude measure of the
total curvature over $[a,b]$

The second result can be interpreted as
optimality property or minimal curvature
property. It means if one considers any
other interpolant, it will oscillate at least
as much as a spline.

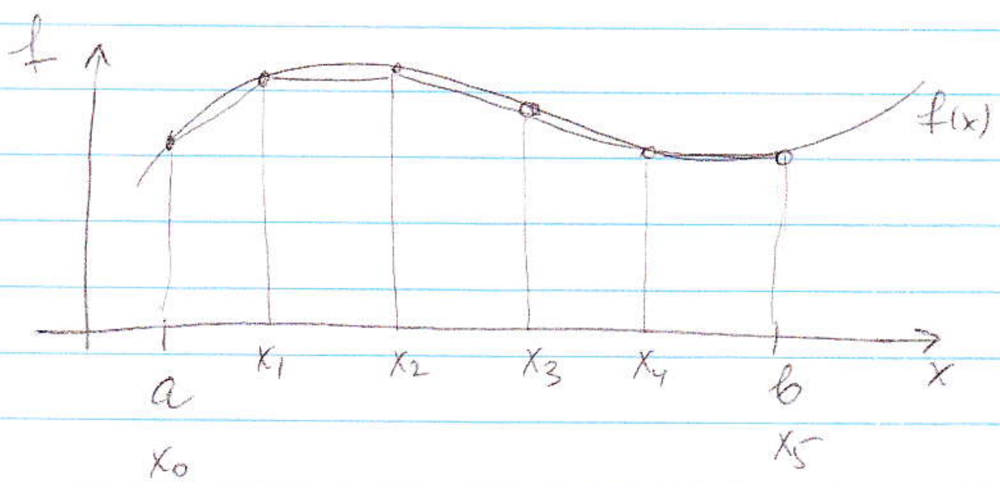
Numerical Integration

Newton-Côtes formulas

Basic idea:
$$\int_a^b f(x)\,dx \sim \sum_{i=0}^{n} c_i f(x_i)$$

For now assume $x_i = a + ih$, $h = \frac{b-a}{n}$

Trapezoidal rule



$$T(h) = h\left(\frac{f(x_0)+f(x_1)}{2}\right) + \dots + h\left(\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$\boxed{T(h) = h\left(\tfrac{1}{2}f(x_0) + f(x_1) + \dots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\right)}$$

trapezoid rule

$\underline{\underline{Ex}}$ $\displaystyle\int_0^1 e^{-x^2}dx = 0.746824\ldots$

h	$T(h)$	error	error / h^2
1	0.683940	0.062884	0.062884
0.5	0.731370	0.015454	0.061816
0.25	0.742984	0.003840	0.06144
0.125	0.745866	0.000958	0.061312

Trapezoid rule is 2^{nd} order accurate

<u>Local error analysis</u> (trapezoid rule)

$$\underbrace{\int_a^{a+h} f(x)dx}_{\text{exact}} = \underbrace{h \cdot \frac{f(a)+f(a+h)}{2}}_{\text{approximation}} \underbrace{- \frac{h^3}{12} f''(\hat{\xi})}_{\text{error}}$$

where $\hat{\xi}$ is between a and $a+h$: $\hat{\xi} \in [a, a+h]$.

$\underline{\underline{Pf}}$

$$f(x) = p_1(x) + \frac{f''(\xi)}{2!}(x-x_0)(x-x_1)$$

$x_0 = a$, $x_1 = a+h$

$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$: linear interpolating polynomial

$$p_1(x) = f(a) + \frac{f(a+h) - f(a)}{h}(x-a)$$

$$\kappa(x) = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}} \approx |f''(x)|$$

$\Rightarrow \int_a^b [f''(x)]^2\,dx$ is a crude measure of the total curvature over an interval.

$\int_a^b [S''(x)]^2\,dx \le \int_a^b [f''(x)]^2\,dx$: any smooth interpolating function must have a total curvature at least as large as that of the cubic spline (natural or clamped)

$S''(x_0) = S''(x_n) = 0$

$S'(x_0) = f'(x_0)$

$S'(x_n) = f'(x_n)$