4/16/2010

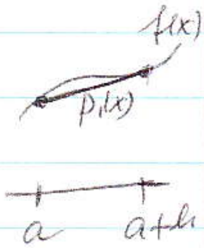
Trapezoid Rule (Cont'd)

Local error analysis

$$\int_a^{a+h} f(x)\,dx = h\,\frac{f(a)+f(a+h)}{2} - \frac{h^3}{12} f''(\xi^1)$$

where ξ^1 is some point on $[a, a+h]$.

exact — approximation — error



Proof

$$f(x) = p_1(x) + \frac{f''(\xi)}{2!}(x-a)(x-(a+h))$$

$$p_1(x) = f(a) + \frac{f(a+h)-f(a)}{h}\cdot(x-a)$$

$$\int_a^{a+h} f(x)\,dx = \int_a^{a+h} p_1(x)\,dx + \int_a^{a+h} \frac{f''(\xi)}{2!}(x-a)(x-(a+h))\,dx$$

$$\int_a^{a+h} p_1(x)\,dx = \int_a^{a+h}\left(f(a) + \frac{f(a+h)-f(a)}{h}(x-a)\right)dx =$$

$$= f(a)\cdot h + \frac{f(a+h)-f(a)}{h}\cdot \left.\frac{(x-a)^2}{2}\right|_{x=a}^{x=a+h} =$$

$$= f(a)\cdot h + \frac{f(a+h)-f(a)}{h}\cdot\frac{h^2}{2} =$$

$$= f(a)\cdot h + \frac{h}{2}\left[f(a+h) - f(a)\right] = \frac{h}{2}\left[f(a) + f(a+h)\right]$$

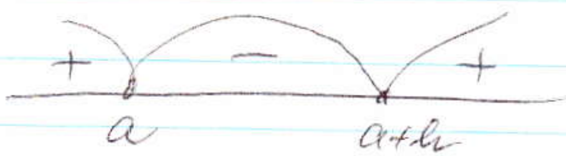
In the error term $\frac{f''(\xi)}{2!}(x-a)(x-(a+h))$,
ξ is actually a function of x, i.e. changing
x will change $\xi(x)$.

$$\int_a^{a+h} \frac{f''(\xi)}{2!}(x-a)(x-(a+h))\,dx = \left|\begin{matrix}\text{generalized}\\ \text{MVT}\end{matrix}\right|$$

$$= \frac{f''(\hat{\xi})}{2}\int_a^{a+h} \underbrace{(x-a)(x-(a+h))}_{\neq 0 \text{ for } a<x<a+h}\,dx \ominus$$

where $\hat{\xi}$ is between a and $a+h$

and is of the same sign (<0)



$p = (x-a)(x-(a+h))$

def $s = \frac{x-a}{h}$, $\frac{x-(a+h)}{h} = s-1$

$ds = \frac{1}{h}dx$ $\qquad x=a \Rightarrow s=0$

$\qquad x=a+h \Rightarrow s=1$

$$\ominus \frac{f''(\hat{\xi})}{2}\int_0^1 h\cdot s \cdot h(s-1)\,h\,ds = \frac{h^3}{2}f''(\hat{\xi})\int_0^1 s(s-1)\,ds$$

$$= \frac{h^3}{2}f''(\hat{\xi})\int_0^1 (s^2-s)\,ds = \frac{h^3}{2}f''(\hat{\xi})\underbrace{\left(\frac{s^3}{3}-\frac{s^2}{2}\right)\Big|_0^1}_{=-\frac{1}{6}} =$$

$$= -\frac{h^3}{12}f''(\hat{\xi})$$

QE

Generalized MVT

1. If $f(x)$ and $g(x)$ are continuous on $[a,b]$, $g(x) \geq 0$, then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

2. Let $f(x)$ be continuous and $x_1, \dots, x_n \in [a,b]$. Then there exists $\xi \in [a,b]$ such that

$$\sum_{i=1}^{n} f(x_i) = n f(\xi)$$

or $$f(\xi) = \frac{f(x_1) + \dots + f(x_n)}{n}$$

Pf Denote by x_m and x_M the values at which $f(x)$ attains its min and max values, i.e.

$$f(x_m) = \min\{f(x),\ x \in [a,b]\}$$

$$f(x_M) = \max\{f(x),\ x \in [a,b]\}$$

Note $x_m, x_M \in [a,b]$.

Define $h(x) = f(x)\int_a^b g(x)\,dx$

Then since $f(x_m) \leq f(x) \leq f(x_M)$

$$f(x_m)g(x) \leq f(x)g(x) \leq f(x_M)g(x)$$

Integrate $\int_a^b$:

$$\int_a^b f(x_m)g(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^b f(x_M)g(x)dx$$

$$\underbrace{f(x_m)\int_a^b g(x)dx}_{h(x_m)} \le \int_a^b f(x)g(x)dx \le \underbrace{f(x_M)\int_a^b g(x)dx}_{h(x_M)}$$

By Intermediate Value thm, there is $\xi \in [a,b]$ such that

$$h(\xi) = \int_a^b f(x)g(x)dx$$

but $h(\xi) = f(\xi)\int_a^b g(x)dx$

$\underline{\underline{\text{ok}}}$

2,

$$\min\{n f(x),\ x \le a \le b\} \le \sum_{i=1}^{n} f(x_i) \le \max\{n f(x),\ x \le a \le b\}$$

$g(x) = n f(x)$ By Intermediate Value thm, there is $\xi \in [a,b]$ such that

$$\underbrace{n f(\xi)}_{g(\xi)} = \sum_{i=1}^{n} f(x_i)$$

Global error estimate (trapezoid rule)

$$\int_a^b f(x)\,dx = T(h) - \frac{f''(\hat{\xi})}{12} h^2(b-a), \quad \hat{\xi} \in [a,b]$$

$$T(h) = h\left(\tfrac{1}{2} f(x_0) + f(x_1) + \ldots + f(x_{n-1}) + \tfrac{1}{2} f(x_n)\right)$$

$$x_i = a + ih \qquad h = \frac{b-a}{n}$$

Pf

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx =$$

$$= \sum_{i=0}^{n-1} \left(h\,\frac{f(x_i) + f(x_{i+1})}{2} - \frac{h^3}{12} f''(\xi_i) \right) =$$

$$= T(h) - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) \underset{\text{MVT}}{\overset{\text{generalized}}{=}}$$

$$= T(h) - \frac{h^3}{12} \underbrace{n}_{= \frac{b-a}{h}} f''(\hat{\xi}) = T(h) - \frac{h^2}{12}(b-a) f''(\hat{\xi}).$$

QE