Alternative derivation (Method of undetermined coefficients)

$$\int_0^h f(x)\,dx \sim c_0 f(0) + c_1 f(h)$$

Find values c_0 and c_1 such that integration formula is exact for polynomials of degree ≤ 1 (degree 0 and 1)

$f(x) = 1$ $\quad \int_0^h 1\,dx = x\Big|_0^h = \boxed{h = c_0 + c_1}$

$f(x) = x$ $\quad \int_0^h x\,dx = \frac{x^2}{2}\Big|_0^h = \boxed{\frac{h^2}{2} = \cancel{c_0 0} + c_1 h}$

$$\left.\begin{array}{l} h = c_0 + c_1 \\ \frac{h}{2} = c_1 \end{array}\right\} \Rightarrow c_0 = \frac{h}{2} \Rightarrow \int_0^h f(x)\,dx \sim \frac{h}{2} f(0) + \frac{h}{2} f(h)$$

$$= h\left(\frac{f(0) + f(h)}{2}\right)$$

Note

$f(x) = x^2$ $\quad \int_0^h x^2\,dx = \frac{h^3}{3} \neq \cancel{\frac{h}{2} 0} + \frac{h}{2} \cdot h^2 = \frac{h^3}{2}$

$$\int_0^h f(x)\,dx \neq c_0 f(0) + c_1 f(h)$$

Thus, the trapezoid rule is exact for polynomials of degree ≤ 1. Trapezoid rule has degree of precision $r = 1$.

Asymptotic expansion

$\cdot 4 \,/ \rightarrow \quad T(h) = \int_a^b f(x)\,dx + C_2 h^2 + C_4 h^4 + C_6 h^6 + \dots$

since error $\sim Ch^2$ (cumulative error)

Richardson extrapolation (Romberg's method)

$$T(2h) = \int_a^b f(x)\,dx + C_2(2h)^2 + C_4(2h)^4 + C_6(2h)^6 + \dots$$

$$\rightarrow \quad T(2h) = \int_a^b f(x)\,dx + 4C_2 h^2 + 16 C_4 h^4 + 64 C_6 h^6 + \dots$$

Multiply expression with $T(h)$ by 4 and subtract $T(2h)$ from $4T(h)$

$$4T(h) - T(2h) = 3\int_a^b f(x)\,dx - 12 C_4 h^4 - 60 C_6 h^6 + \dots$$

$$\frac{4T(h) - T(2h)}{3} = \int_a^b f(x)\,dx - 4 C_4 h^4 - 20 C_6 h^6 + \dots$$

Define $R_1(h) = \dfrac{4T(h) - T(2h)}{3}$; 4th order accurate

$16 \,/ \quad R_1(h) = \int_a^b f(x)\,dx + \tilde{C}_4 h^4 + \tilde{C}_6 h^6 + \dots$

$$R_1(2h) = \int_a^b f(x)\,dx + \underbrace{\tilde{C}_4 (2h)^4}_{= 16h^4 C_4} + \tilde{\tilde{C}}_6 (2h)^6 + \dots$$

$$\frac{16-64}{15} = \frac{-48}{15} = \frac{-16}{5}$$

$$\frac{16 R_1(h) - R_1(2h)}{15} = \int_a^b f(x)dx - \frac{16}{5} \tilde{C}_6 h^6 + \dots$$

Define

$$R_2(h) = \frac{16 R_1(h) - R_1(2h)}{15} : \underline{6^{th} \text{ order accurate}}$$

<u>Note</u>

Let $R_0(h) = T(h)$, then

$$R_1(h) = \frac{4 R_0(h) - R_0(2h)}{3} = R_0(h) + \frac{R_0(h) - R_0(2h)}{3}$$

$$R_2(h) = \frac{16 R_1(h) - R_1(2h)}{15} = R_1(h) + \frac{R_1(h) - R_1(2h)}{15}$$

$64 = 4^3$

$64 - 1 = 63$

$$R_3(h) = \frac{64 R_2(h) - R_2(2h)}{63} = R_2(h) + \frac{R_2(h) - R_2(2h)}{63}$$

— — — —

$$R_n(h) = \frac{4^n R_{n-1}(h) - R_{n-1}(2h)}{4^n - 1}$$

table

$$
\begin{tikzcd}
2h & R_0(2h) \arrow[dr, "\frac{\Delta}{3}"] & & \\
h & R_0(h) \arrow[r] \arrow[dr, "\frac{\Delta}{3}"] & R_1(h) \arrow[dr, "\frac{\Delta}{15}"] & \\
\frac{h}{2} & R_0\left(\frac{h}{2}\right) \arrow[r] & R_1\left(\frac{h}{2}\right) \arrow[r] & R_2\left(\frac{h}{2}\right)
\end{tikzcd}
$$

Note

down a column: decreasing h, fixed order of accuracy

across a row: fixed h, increasing order of accuracy

Ex $\int_0^1 e^{-x^2}\,dx = 0.74682413\ldots$

h	trapezoid $R_0(h)$	simpson $R_1(h)$	boole's $R_2(h)$	$R_3(h)$
1	0.683940			
0.5	0.731370	0.747180		
0.25	0.742984	0.7468553	0.7468336	
0.125	0.745866	0.7468266	0.7468246	0.7468244

Note Value in the last column $(R_3(0.125))$ is obtained by using a difference divided by 63, i.e. $\frac{\Delta}{63}$.

Simpson's rule

$$R_1(h) = R_0(h) + \frac{R_0(h) - R_0(2h)}{3}$$

or

$$R_1(h) = T(h) + \frac{T(h) - T(2h)}{3}$$