$$\Rightarrow \quad 0 = e^{(n+1)}(\xi) = f^{(n+1)}(\xi) - g^{(n+1)}(\xi)$$

$$g^{(n+1)}(\xi) = \cancel{p_n^{(n+1)}(\xi)}^{\,0} + f[x_0, x_1, \dots, x_n, x] \cdot (n+1)!$$

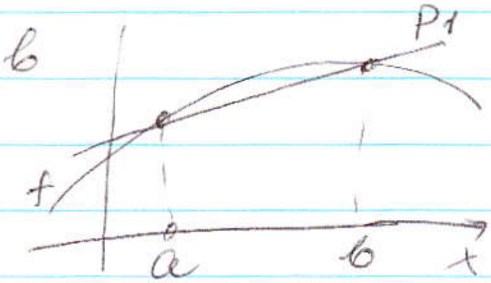
$$\Rightarrow f^{(n+1)}(\xi) - f[x_0, x_1, \dots, x_n, x] \cdot (n+1)! = 0$$

$$\Rightarrow f[x_0, x_1, \dots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Substitute $f[x_0, x_1, \dots, x_n, x]$ into $(*)$ to get

$$\boxed{f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)}$$

<u>Application</u> : $n=1$, $x_0 = a$, $x_1 = b$



Claim

If $|f''(x)| \le M$ for all $x \in [a, b]$, then

$$\left(1 - \frac{1}{x}\right)$$

$$\|f(x) - p_1(x)\|_\infty = \sup_{x \in [a,b]} |f(x) - p_1(x)| \le \frac{M}{8}(b-a)^2$$

Pf

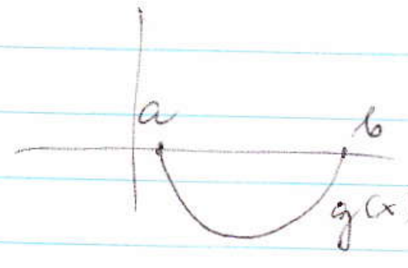
$$|f(x) - p_1(x)| = \left|\frac{f''(\xi)}{2!}(x-a)(x-b)\right| \le \frac{M}{2}|(x-a)(x-b)|$$

$$g(x) = (x-a)(x-b) = x^2 - (a+b)x + ab$$

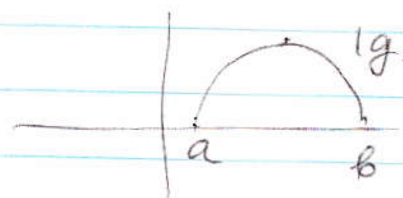
$$g'(x) = 2x - (a+b) \Rightarrow g'(x) = 0 \text{ at } \bar{x} = \frac{a+b}{2}$$

$$g''(x) = 2 > 0 \Rightarrow g(x) \text{ has local min at } x = \bar{x}$$

$$g(a) = 0, \quad g(b) = 0$$



$\Rightarrow |g(x)|$ has local max (and global max) at $x = \bar{x}$



$$|g(\bar{x})| = \left| \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right) \right| =$$

$$= \left| \left(\frac{b}{2} - \frac{a}{2}\right)\left(\frac{a}{2} - \frac{b}{2}\right) \right| = \frac{(b-a)^2}{4}.$$

$$\Rightarrow |g(x)| \le \frac{(b-a)^2}{4}$$

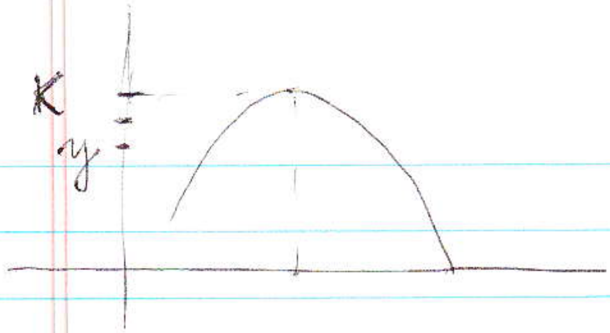
$$\Rightarrow |f(x) - p_1(x)| \le \frac{M}{2} |(x-a)(x-b)| \le \frac{M}{2} \cdot \frac{(b-a)^2}{4} = \frac{M}{8}(b-a)^2$$

true for any $x \in [a,b]$

$$\Rightarrow \|f - p_1\|_\infty = \sup_{[a,b]} |f - p_1| \le \frac{M}{8}(b-a)^2$$

$$\| \quad = \max_{a \le x \le b} |f(x) - p_1(x)|$$

$K = \sup_{\text{on } [a,b]} f$ if for any $x \in [a,b)$



$K = \sup f$

given $\forall \varepsilon > 0$

$\forall y < K \quad \exists x \in [a, b]: \quad f(x) \le K$

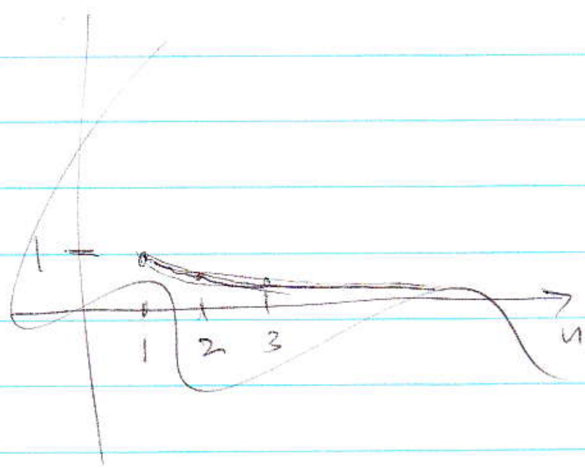
$\cancel{f(n) = \frac{1}{n}, \quad n = 1, 2, 3, \dots}$

$f(n) = 1 - \frac{1}{n}, \quad n = 1, 2, 3 \dots$

as $n \nearrow$, $f(n) \to 1$

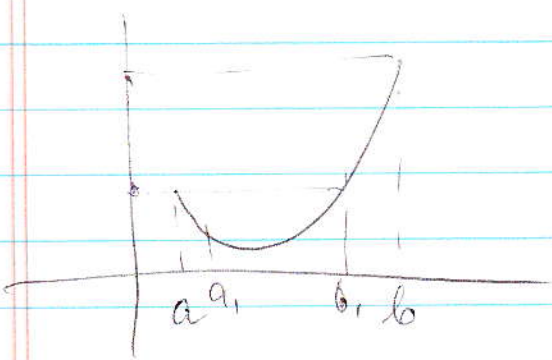
$1 = \sup\{1 - \frac{1}{n}\}$

$n = k: \quad f(k) = \frac{1}{k}$

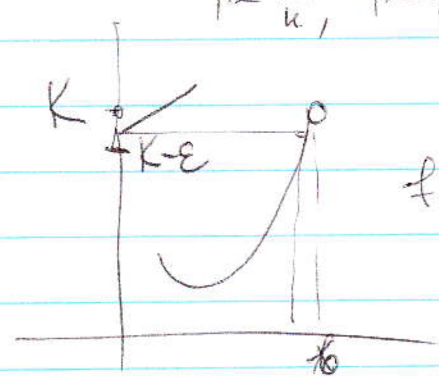


$n = k+1 \qquad \frac{1}{k}, \frac{1}{k+1}, k$

$1 - \frac{1}{k}, \ 1 - \frac{1}{k+1}, \ 1$



ε



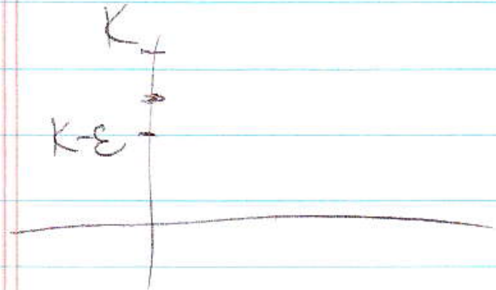
$f(K - \varepsilon) \le K$

$x \in [a, b]$ $\qquad$ $x \in [a, b)$

$K = \max f \quad \exists \xi \in [a, b]: \ f(\xi) = K$

Def K is a sup of $f(x)$ on $[a,b]$ if for any $\varepsilon > 0$ $\exists\, x \in \cancel{[a,b]}$ such that

$$\cancel{f(K-\varepsilon)}\ f(K-\varepsilon) \le f(x) < K$$



if K is max f then

$\exists\, x \in [a,b] : f(x) = K$

Ex $f(x) = \dfrac{1}{x}$, $x_0 = a = 1$, $x_1 = b = 2$

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$$

$$M = \sup_{1 \le x \le 2} |f''(x)| = \max_{1 \le x \le 2} \left|\frac{2}{x^3}\right| = 2 \Rightarrow M = 2$$

$$\|f - p_1\|_\infty \le \frac{M}{8}(b-a)^2 = \frac{2}{8} = \frac{1}{4}$$

$$\|f - p_1\|_\infty = \max_{a \le x \le b} |f(x) - p_1(x)|$$

Questions

1. Given f, $[-1, 1]$, what is the best choice of interpolating $x_0, x_1, \dots, x_n$?

2. $p_n \to f$ for all $x \in [a, b]$ as $n \to \infty$?