4/21/2010

#3 $\int_a^b f(x)\,dx$

$x_0 = a, \quad x_1 = a+h+\varepsilon, \quad x_2 = a+2h = b$

$$f(x) = p_2(x) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{exact}} = \underbrace{\int_a^b p_2(x)\,dx}_{\text{gives quadrature}} + \underbrace{\int_a^b \frac{f'''(\xi(x))}{3!} \overbrace{(x-x_0)(x-x_1)(x-x_2)}^{\text{changes its sign at } x_1 \in (a,b)}\,dx}_{I}$$

Since $g(x) = (x-x_0)(x-x_1)(x-x_2)$ changes its sign inside $[a,b]$ we cannot apply generalized MVT directly:

$$\int_a^b f(x)\underbrace{g(x)}_{>0}\,dx = f(\xi)\int_a^b g(x)\,dx$$

Use integration by parts in I

$u = (x-x_0)(x-x_1)(x-x_2) \qquad dV = \frac{f'''(\xi(x))}{3!}$

$du = \left(\text{polynomial of degree 2}\right)dx$ with no roots inside $[a,b]$ $\qquad V = \frac{f''(\hat{\xi})}{3!}$

Then use generalized MVT

Integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

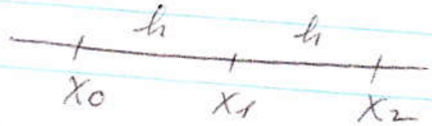
Ex $\displaystyle\int_0^{\pi} x \sin x \, dx = \left| \begin{matrix} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{matrix} \right| =$

$$= -x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx = \pi + \sin x \Big|_0^{\pi} = \pi$$

Simpson's Rule

$$R_1(h) = T(h) + \frac{T(h) - T(2h)}{3}$$

local form



$$T(h) = h\left(\tfrac{1}{2} f(x_0) + f(x_1) + \tfrac{1}{2} f(x_2)\right)$$

$$T(2h) = 2h\left(\tfrac{1}{2} f(x_0) + \tfrac{1}{2} f(x_2)\right) = h f(x_0) + h f(x_2)$$

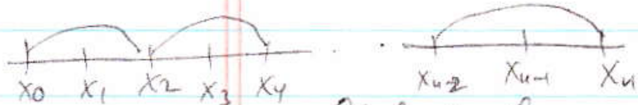
$$R_1(h) = h\left(\tfrac{1}{2} f(x_0) + f(x_1) + \tfrac{1}{2} f(x_2)\right) +$$

$$+ \tfrac{1}{3}\Big(\tfrac{h}{2} f(x_0) + h f(x_1) + \tfrac{h}{2} f(x_2) - h f(x_0) - h f(x_2)$$

$$= f(x_0)\left(\frac{h}{2} + \frac{h}{6} - \frac{h}{3}\right) + f(x_1)\left(h + \frac{h}{3}\right) +$$

$$+ f(x_2)\left(\frac{h}{2} + \frac{h}{6} - \frac{h}{3}\right) = \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) + \frac{h}{3} f(x_2)$$

$$\Rightarrow \quad R_1(h) = h\left(\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2)\right)$$



Global form (assume n is even)

$$R_1(h) = h\left(\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{2}{3} f(x_2) + \frac{4}{3} f(x_3) + \right.$$

$$\left. + \frac{2}{3} f(x_4) + \ldots + \frac{2}{3} f(x_{n-2}) + \frac{4}{3} f(x_{n-1}) + \frac{1}{3} f(x_n)\right)$$

<u>Error</u>

$$\int_a^b f(x)\,dx = R_1(h) - \frac{f^{(4)}(\hat{\xi})}{180} h^4 (b-a), \quad \hat{\xi} \in [a, b]$$

<u>Ex</u> Use the method of undetermined coefficients to derive Simpson's rule.

$$\int_0^{2h} f(x)\,dx \sim c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

$$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h$$

$$\sim c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

$f(x)=1$ $\quad \int_0^{2h} 1\,dx = \boxed{2h = c_0\cdot 1 + c_1\cdot 1 + c_2\cdot 1}$

$f(x)=x$ $\quad \int_0^{2h} x\,dx = \frac{x^2}{2}\Big|_0^{2h} = 2h^2 = c_0\cdot \cancel{0}^{\,0} + c_1 h + c_2\cdot 2h$

$\boxed{2h = c_1 + 2c_2}$

$f(x)=x^2$ $\quad \int_0^{2h} x^2\,dx = \frac{x^3}{3}\Big|_0^{2h} = \frac{8}{3}h^3 = c_0\cdot\cancel{0}^{\,0} + c_1\cdot h^2 + c_2\cdot 4h^2$

$\boxed{\frac{8}{3}h = c_1 + 4c_2}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2h \\ 2h \\ 8h/3 \end{pmatrix}$$

Solution: $\boxed{c_0 = c_2 = \frac{h}{3}, \quad c_1 = \frac{4h}{3}}$

$\frac{h}{3} + \frac{4h}{3} + \frac{h}{3} = \frac{6h}{3} = 2h$ ✓

$\frac{4h}{3} + 2\cdot\frac{h}{3} = \frac{6h}{3} = 2h$ ✓

$\frac{4h}{3} + 4\cdot\frac{h}{3} = \frac{8h}{3}$ ✓

ok

<u>Note</u>

$f(x)=x^3$ $\quad \int_0^{2h} x^3\,dx = \frac{x^4}{4}\Big|_0^{2h} = \frac{2^4h^4}{2^2} = 4h^4$

$\|$

$c_0 f(0) + c_1 f(h) + c_2 f(2h) = \frac{h}{3}\cdot\cancel{0}^{\,0} + \frac{4h}{3}\cdot h^3 + \frac{h}{3}\,8h^3$

$= \frac{12h^4}{3} = 4h^4$

$$f(x) = x^4 \qquad \int_0^{2h} x^4 \, dx \neq c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

Hence, Simpson's rule is exact for polynomials of degree ≤ 3.

Trapezoid and Simpson's rules are example of Newton-Côtes formulas.

Orthogonal Polynomials

Define the inner product of two functions on $[-1,1]$ as

$$\int_{-1}^{1} f(x) g(x) \, dx = \langle f, g \rangle$$

Properties

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \iff f \equiv 0$

$$\langle f, f \rangle = \| f \|^2$$

$$\| f \| = \sqrt{\langle f, f \rangle} \; : \text{norm of } f$$

2. $\langle f, \alpha h + g \rangle = \alpha \langle f, h \rangle + \langle f, g \rangle$

We say that functions f and g are orthogonal if $\langle f, g \rangle = 0$

$\sin 2t = 2\sin t\cos t$

Ex $\sin\pi x$ and $\cos\pi x$ are orthogonal on $[-1,1]$

$$\int_{-1}^{1} \sin\pi x \cdot \cos\pi x \, dx = \frac{1}{2}\int_{-1}^{1} \sin 2\pi x \, dx = -\frac{1}{2}\frac{1}{2\pi}\cos 2\pi x \Big|_{x=-1}^{1}$$

$= 0$

Ex $\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x \, dx = 0 \Rightarrow 1$ and x are orthogonal

$$\langle 1, x^2\rangle = \int_{-1}^{1} 1\cdot x^2 \, dx \underset{\substack{\text{symmetry} \\ \text{of } x^2}}{=} 2\int_{0}^{1} x^2 \, dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3} \neq 0$$

$\Rightarrow 1$ and x^2 are NOT orthogonal