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Gram-Schmidt orthogonalization method

Given a sequence of linearly independent functions $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$, the Gram-Schmidt orthogonalization method produces a sequence $\{\psi_0, \psi_1, \psi_2, \dots\}$ of mutually orthogonal functions

In particular, given a sequence $\{1, x, x^2, \dots\}$: lin. independ.

Gram-Schmidt process gives a sequence of orthogonal polynomials $\{p_0(x), p_1(x), p_2(x), \dots\}$ called Legendre polynomials

$$\boxed{p_0 = 1} \leftarrow \varphi_0$$

$$p_1 = \underbrace{x}_{\varphi_1} + \alpha_{10} \cdot \underbrace{p_0}_{\varphi_0} \qquad \Big| \cdot p_0$$

We want to find coefficient α_{11} such that p_0 and p_1 are orthogonal

$$\cancelto{0}{\langle p_1, p_0 \rangle} = \langle x, p_0 \rangle + \alpha_{10} \underbrace{\langle p_0, p_0 \rangle}_{\|p_0\|^2}$$

$$\Rightarrow \alpha_{10} = - \frac{\langle x, p_0 \rangle}{\|p_0\|^2} = 0 \quad \Rightarrow \alpha_{10} = 0$$

where

$$\boxed{p_1 = x}$$

$$\langle x, p_0 \rangle = \int_{-1}^{1} x \cdot 1 \, dx = 0$$

$$\| p_0 \|^2 = \langle p_0, p_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2$$

$$p_2 = x^2 + \alpha_{21} p_1 + \alpha_{20} \cdot p_0 \quad \Big| \cdot p_1 \quad \Big| \cdot p_0$$

We want to find α_{21}, α_{20} in such a way that p_2 is orthogonal to both p_1 and p_0

$$\cancelto{0}{\langle p_2, p_1 \rangle} = \langle x^2, p_1 \rangle + \alpha_{21} \underbrace{\langle p_1, p_1 \rangle}_{\| p_1 \|^2} + \alpha_{20} \cancelto{0}{\langle p_0, p_1 \rangle}$$

$$\Rightarrow \alpha_{21} = - \frac{\langle x^2, p_1 \rangle}{\| p_1 \|^2} = 0$$

$$\cancelto{0}{\langle p_2, p_0 \rangle} = \langle x^2, p_0 \rangle + \alpha_{21} \cancelto{0}{\langle p_1, p_0 \rangle} + \alpha_{20} \underbrace{\langle p_0, p_0 \rangle}_{\| p_0 \|^2}$$

$$\Rightarrow \alpha_{20} = - \frac{\langle x^2, p_0 \rangle}{\| p_0 \|^2} = - \frac{2}{3 \cdot 2} = - \frac{1}{3}$$

where

$$\langle x^2, p_1 \rangle = \int_{-1}^{1} x^2 \cdot x \, dx = 0$$

$$\langle x^2, p_0 \rangle = \int_{-1}^{1} x^2 \cdot 1 = 2 \int_{0}^{1} x^2 \, dx = \frac{2}{3}$$

$$\| p_0 \|^2 = 2 \qquad \Rightarrow \alpha_{21} = 0, \quad \alpha_{20} = - \frac{1}{3}$$

$$\Rightarrow \quad p_2(x) = x^2 + 0\cdot p_1 - \frac{1}{3}\cdot p_0 = \boxed{x^2 - \frac{1}{3} = p_2(x)}$$

Summary,

$$p_1 = x + \alpha_{10}\cdot p_0 = x - \frac{\langle x, p_0\rangle}{\|p_0\|^2}\cdot p_0$$

$$p_2(x) = x^2 + \alpha_{21} p_1 + \alpha_{20} p_0$$

$$= x^2 - \frac{\langle x^2, p_1\rangle}{\|p_1\|^2} p_1 - \frac{\langle x^2, p_0\rangle}{\|p_0\|^2} p_0$$

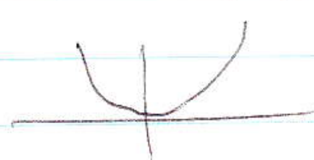
Now

$$p_3(x) = x^3 - \cancelto{0}{\frac{\langle x^3, p_2\rangle}{\|p_2\|^2}} p_2 - \frac{\langle x^3, p_1\rangle}{\|p_1\|^2} p_1 - \cancelto{0}{\frac{\langle x^3, p_0\rangle}{\|p_0\|^2}} p_0$$

$$\langle x^3, p_2\rangle = \int_{-1}^{1} \underbrace{\underbrace{x^3}_{\text{odd}}\underbrace{\left(x^2 - \frac{1}{3}\right)}_{\text{even}}}_{\text{odd}} dx = 0$$

Recall, $f(x)$ is an odd function if $f(-x) = -f(x)$

$f(x)$ is even function if $f(-x) = f(x)$



$$\langle x^3, p_1\rangle = \int_{-1}^{1} x^3 \cdot x \, dx = \int_{-1}^{1} x^4 dx = 2\int_0^1 x^4 dx = 2\frac{x^5}{5}\Big|_0^1 = \frac{2}{5}$$

$$\|p_1\|^2 = \int_{-1}^{1} p_1^2 \, dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$\Rightarrow \frac{\langle x^3, p_1\rangle}{\|p_1\|^2} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}$$

$$\langle x^3, p_0\rangle = \int_{-1}^{1} x^3 \cdot 1 \, dx = 0$$

$$\Rightarrow \boxed{p_3(x) = x^3 - \frac{3}{5}x}$$

Note LEGENDRE

1. $p_n(x)$ is a Legendre polynomial of degree n

2. Any polynomial of degree $\leq n$ can be written

$$q(x) = \sum_{i=0}^{n} c_i p_i(x)$$

Legendre polynomials $\{p_i(x)\}_{i=0}^{n}$ form a basis of the set $\mathcal{P}_n$ of polynomials of degree $\leq n$

Gaussian Quadrature

1. Legendre polynomials $p_n(x)$ have n distinct roots in $(-1,1)$, say x_i, $i=1,\dots,n$

2. There exist coefficients c_i, $i=1,\dots,n$ such that
$$\int_{-1}^{1} f(x)\,dx \sim \sum_{i=1}^{n} c_i f(x_i)$$
is <u>exact</u> for polynomials of degree $\leq 2n-1$.

$(n \geq 2)$

<u>Ex</u>
$$\int_0^1 e^{-x^2}dx = \left| \begin{matrix} t=2x-1 \\ dt = 2\,dx \end{matrix} \right| = \int_{-1}^{1} e^{-\frac{(t+1)^2}{4}} \cdot \frac{dt}{2} =$$

$t=ax+b$

$x = \frac{t+1}{2}$

$x=0 \Rightarrow t=-1 \Rightarrow b=-1$

$x=1 \Rightarrow t=1 \Rightarrow 1=a+\underset{-1}{\underset{\shortparallel}{b}} \rightarrow a=2$

$$= \int_{-1}^{1} \frac{e^{-\frac{(t+1)^2}{4}}}{2}\,dt$$

n	G_n
2	0.746595
3	0.746816
4	0.746824

This is much better than with trapezoid or Simpson's rule alternative definition of Legendre polynomials:

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x^2)^n\right] \qquad n \geq 1$$

$P_0(x) = 1 \qquad \deg P_n = n$

$P_n(1) = 1$ – normalization condition used here

$$P_1(x) = \frac{(-1)^1}{2^1 \cdot 1!} \underbrace{\frac{d}{dx}(1-x^2)}_{-2x} = x$$