<u>Claim</u>

1). Legendre polynomial $P_n(x)$ has n distinct roots in $(-1,1)$, say x_i, $i=1,2,..,n$

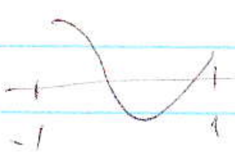
2). There exist constants c_i, $i=1,2,..,n$ such that

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{i=1}^{n} c_i f(x_i) \qquad \text{Gauss quadrature}$$

is exact for polynomials of degree $\leq 2n-1$, $n \geq 2$

<u>Pf of 1)</u>

For $n \geq 1$, $0 = \langle P_n, P_0 \rangle = \int_{-1}^{1} P_n(x)\,dx$



This implies that $P_n(x)$ has at least one root of odd multiplicity in $(-1,1)$. Let x_i be points on $(-1,1)$ at which $P_n(x)$ changes its sign, $i=1,2,.., j \leq n$:

$-1 < x_1 < x_2 < .. < x_j < 1$, we know $j \leq n$.

Form the function $g(x) = \prod_{i=1}^{j} (x - x_i)$

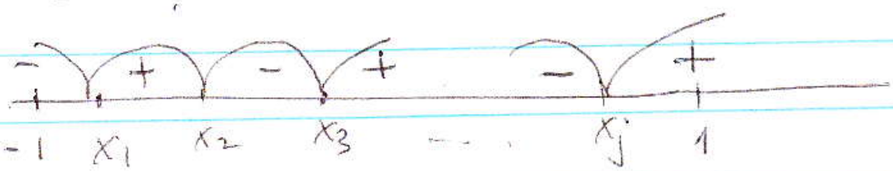
$= (x-x_1)(x-x_2)\ldots(x-x_j)$

product of j terms

$\sum_{i=1}^{j} (x-x_i) =$

$= (x-x_1) + (x-x_2) + \ldots + (x-x_j)$

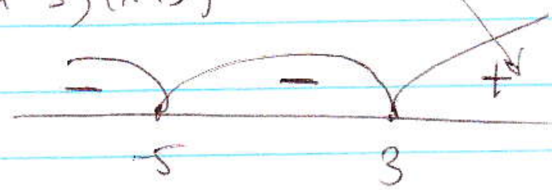
$$q(x) = (x-x_1)(x-x_2)\ldots(x-x_j)$$



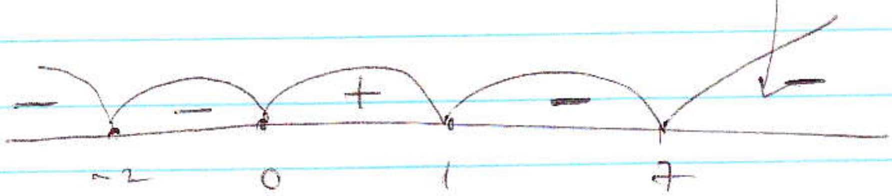
Aside

$(x-x_1)(x-x_2)\ldots$

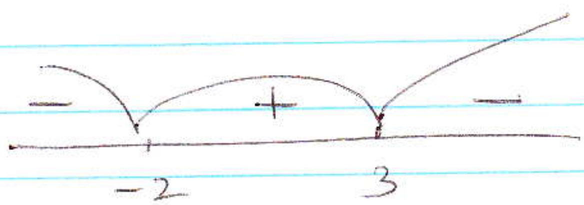
$$f(x) = \overset{>0}{2}(x-3)(x+5)^2$$



$$f(x) = \overset{<0}{-}(x-1)(x+2)^4\, x^3 (x-7)^2$$

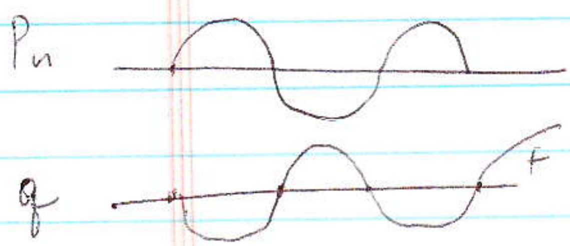


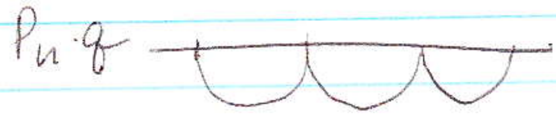
$$f(x) = (3-x)(x+2) = -(x-3)(x+2)$$



Sign of $g(x)$ also changes at points x_i, $i=1,\ldots,j$.

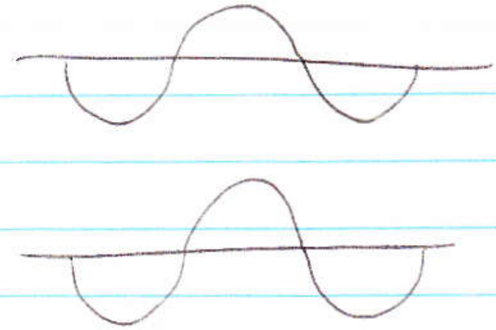
Then $\langle P_n, q\rangle = \int_{-1}^{1} P_n(x)\, q(x)\, dx \neq 0$



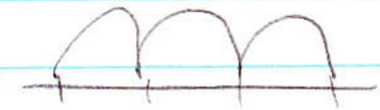


or P_n

q



$P_n q$



since $P_n(x)$ and $q(x)$ have either positive or negative sign over each interval

Since $\langle P_n, q\rangle \neq 0 \Rightarrow P_n(x)$ and $q(x)$ are not orthogonal.

But $\langle P_n, q\rangle \neq 0 \Rightarrow \deg q \geq n$
since otherwise if $\deg q < n \Rightarrow \langle P_n, q\rangle = 0$
since $\{P_0, P_1, \dots, P_n\}$ form a basis in the set $\mathcal{P}_n$ of polynomials of degree $\leq n$

~~p: polynomial of degree $\leq n$~~

~~$p = \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_n P_n$~~

~~$\langle p, P_0\rangle$~~

if $\deg q = j < n \Rightarrow q = \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_j P_j \ , \ j < n$

$$\langle q, P_n\rangle = \alpha_0 \cancelto{0}{\langle P_0, P_n\rangle} + \alpha_1 \cancelto{0}{\langle P_1, P_n\rangle} + \dots + \alpha_j \cancelto{0}{\langle P_j, P_n\rangle}$$

Thus, $j \le n$ and $j \ge n \Rightarrow j = n$, i.e. all roots of $P_n(x)$ are inside $(-1,1)$.

<u>Note</u>: all these roots are simple roots

<u>Proof 2</u>. Assume that $f(x)$ is a polynomial of degree $\le 2n-1$.

<u>Case 1</u> degree $f \le n-1$

$$f(x) = \sum_{j=1}^{n} f(x_j)\, l_j(x)$$

where $l_j(x)$ are Lagrange polynomials based on points $x_1, x_2, \dots, x_n$.

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \sum_{j=1}^{n} f(x_j)\, l_j(x)\,dx =$$

$$= \sum_{j=1}^{n} f(x_j) \underbrace{\int_{-1}^{1} l_j(x)\,dx}_{= c_j} = \sum_{j=1}^{n} c_j f(x_j) \ : \text{exact}$$

<u>Note</u> $c_j = \int_{-1}^{1} l_j(x)\,dx$

Case 2 $\quad$ degree $f \leq 2n-1$

$f = q P_n + r \qquad$ q: quotient, $\deg q \leq n-1$
$\qquad\qquad\qquad\qquad$ r: remainder, $\deg r \leq n-1$

$x^{2n-1} = x^n \cdot x^{n-1}$

$x^{2n-1} + \dots = x^n (x^{n-1} + \dots)$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \cancel{\underbrace{q(x)}_{\deg q \leq n-1} P_n(x)\,dx}^{\,0} + \int_{-1}^{1} \underbrace{r(x)\,dx}_{\deg r \leq n-1} =$$

$$= \cancel{\langle q, P_n \rangle}^{\,0} + \sum_{j=1}^{n} c_j \cdot r(x_j) \; = \quad \text{using case 1}$$

case 1

But $f(x_j) = q(x_j) \underbrace{\cancel{P_n(x_j)}^{\,0}}_{\substack{\text{as } x_j \text{ is} \\ \text{a root of } P_n}} + r(x_j)$

$= \sum_{j=1}^{n} c_j \cdot f(x_j)$: exact

$\Rightarrow \int_{-1}^{1} f(x)\,dx = \sum_{j=1}^{n} c_j f(x_j)$ $\quad$ exact for $f(x)$: polynomial of degree $\leq 2n-1$

Ex Derive 3-point Gauss quadrature rule.

$$\int_{-1}^{1} f(x)dx \sim C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

$P_3(x) = x^3 - \frac{3}{5}x = x\left(x^2 - \frac{3}{5}\right)$: 3rd order Legendre polynomial

Take $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$

$C_j = \int_{-1}^{1} l_j(x)dx$, $l_j(x)$: Lagrange polynomials

Another way to compute C_j is to use the method of undetermined coefficients.

$f(x) = 1$ $\quad \int_{-1}^{1} 1\,dx = 2 = C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$

$$\boxed{2 = C_1 + C_2 + C_3}$$

$f(x) = x$ $\quad \int_{-1}^{1} x\,dx = \boxed{0 = C_1\left(-\sqrt{\frac{3}{5}}\right) + C_2 \cdot 0 + C_3 \cdot \sqrt{\frac{3}{5}}}$

$f(x) = x^2$ $\quad \int_{-1}^{1} x^2 dx = 2\int_{0}^{1} x^2 dx = \boxed{\frac{2}{3} = C_1 \cdot \frac{3}{5} + C_2 \cdot 0 + C_3 \cdot \frac{3}{5}}$

$\Rightarrow C_1 = C_3 = \frac{5}{9}$, $\quad C_2 = \frac{8}{9}$

3-point Gauss quadrature is

$$\int_{-1}^{1} f(x)\,dx \sim \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right)$$

exact for polynomials ≤ 5 (2.3-1)