Error in Gauss integration

$$\underbrace{\int_{-1}^{1} f(x)\,dx}_{\text{exact}} = \underbrace{\sum_{j=1}^{n} c_j f(x_j)}_{\text{approximation}} + \underbrace{E_n(f)}_{\text{error}}$$

where

$$E_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)\left[(2n)!\right]^2} \cdot \frac{f^{(2n)}(\xi)}{(2n)!}$$

$n=2$: $\quad E_2 f = \dfrac{f^{(4)}(\xi)}{135}$

$n=3$: $\quad E_3 f = \dfrac{f^{(6)}(\xi)}{15,750}$

Singular Integrals

Ex $\quad \int_0^\infty f(x)\, e^{-x}\, dx$

Method 1 (truncation of the domain)

$$\int_0^\infty f(x) e^{-x}\,dx = \underbrace{\int_0^L f(x) e^{-x}\,dx}_{\text{I}} + \underbrace{\int_L^\infty f(x) e^{-x}\,dx}_{\text{II}}$$

I ~ use trapezoid, Simpson's rule or Gauss quadrature

$$\left| \text{II} \right| = \left| \int_L^\infty f(x)\, e^{-x}\, dx \right| \le \int_L^\infty |f(x)| \cdot e^{-x}\, dx \le$$

$$\le \|f\|_\infty \cdot \int_L^\infty e^{-x}\, dx = \|f\|_\infty \left(-e^{-x}\right)\Big|_{x=L}^{\infty} =$$

$$= \|f\|_\infty \cdot e^{-L}$$

Choose L large enough to make $\|f\|_\infty \cdot e^{-L}$ smaller than some given tolerance ε, then integrate I.

<u>Method 2</u> (mapping to a finite interval)

Let $u = e^{-x}$

$$\int_0^\infty f(x) e^{-x}\, dx = \left| \begin{array}{l} u = e^{-x} \qquad x = -\ln u \\ du = -e^{-x} dx = -u\, dx \\ x=0 \Rightarrow u=1 \\ x=\infty \Rightarrow u=0 \end{array} \right| =$$

$$= \int_1^0 f(-\ln u)\,(-du) = \int_0^1 f(-\ln u)\, du$$

Use trapezoid, Simpson, Gauss, ...

Method 3 (Gauss-Laguerre Quadrature)

Define a new inner product:

$$\langle f,g\rangle = \int_0^\infty f(x)g(x)\underbrace{e^{-x}}_{\text{weight function}}\,dx$$

Check

1. $\langle f,f\rangle \geq 0$ and $\langle f,f\rangle = 0 \iff f=0$
2. $\langle f, \alpha g+h\rangle = \alpha\langle f,g\rangle + \langle f,h\rangle$

The Laguerre polynomials are orthogonal with respect to this inner product. They can be obtained using Gram-Schmidt orthogonalization method applied to $\{1, x, x^2, \dots\}$

$L_0(x) = 1$

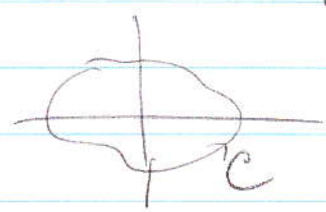
$L_1(x) = x-1$

$L_2(x) = x^2 - 4x + 2$

$\dots$

Aside

$$L_n(z) = \frac{1}{2\pi i}\oint_C \frac{e^{-zt/(1-t)}}{(1-t)t^{n+1}}\,dt$$



Let $x_1, x_2, \dots, x_n$ be roots of polynomial $L_n(x)$.

$c_j = \int_0^\infty \ell_j(x)e^{-x}dx$. Then

$$\int_0^\infty f(x)e^{-x}dx \sim \sum_{j=1}^{n} c_j f(x_j)$$ Gauss-Laguerre quadrature

where x_j, $j=1,\dots,n$ are roots of $L_n(x)$.
This formula is exact for polynomials of degree $\leq 2n-1$.

Ex the 2-point Gauss-Laguerre quadrature is

$$\int_0^\infty f(x)e^{-x}\,dx \sim \left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right) f(2-\sqrt{2}) + \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right) f(2+\sqrt{2})$$

$$\sim 0.854 \cdot f(0.586) + 0.146 \cdot f(3.414)$$

IVP for ODEs

Find a function $y(t)$ that satisfies the equation $y' = f(y)$ subject to initial condition $y(0) = y_0$.

Ex

(a) $y' = y$, $y(0) = 1 \Rightarrow y(t) = e^t$

$$\frac{dy}{dt} = y$$

$$\frac{dy}{y} = dt \Rightarrow \ln|y| = t + \tilde{C}$$

$$y = Ce^t$$

$$y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1$$

(b) $y' = y^2, \quad y(0) = 1 \quad \Rightarrow \quad y(t) = \frac{1}{1-t}$ - blows up at $t=1$

(c) $y' = y^{1/2}, \quad y(0) = 0$

$$\frac{dy}{dt} = \sqrt{y}$$

$$y(t) = \begin{cases} \frac{t^2}{4} \\ 0 \end{cases} \quad \text{: 2 solutions}$$

$$\frac{dy}{\sqrt{y}} = dt \quad \text{assuming } y \neq 0$$

$$\Rightarrow y(t) = \frac{t^2}{4}$$

Is $y \equiv 0$ a solution? Yes

(d) $y' = \sin y, \quad y(0) = 1 \quad \Rightarrow y(t) - ?$

<u>Def</u>

An initial value problem is <u>well-posed</u> if the following properties are satisfied.

1. A solution <u>exists</u>

2. The solution is <u>unique</u>

3. The solution <u>depends continuously</u> on f and and y_0 (small change in f and/or y_0 will result in small change in solution y)

$\frac{dy}{dt} = f(t,y)$

<u>Thm</u> If $|f_y(y)|$ is bounded for $y \in D$, then the initial value problem $y' = f(y)$ subject to $y(0) = y_0$ is locally well posed for all $y_0 \in D$.