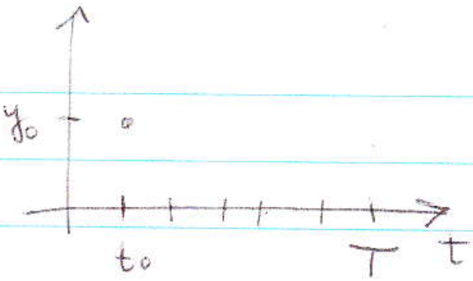
## Euler's method




$h = \Delta t$ = step size

$t_n = nh$ if $t_0 = 0$

$\frac{dy}{dt} = f(t,y)$

$y = y(t)$

$y(t_0) = y_0$

$y(t)$ - ?

$u_n \sim y(t_n)$

approximation exact

$y' = \frac{dy}{dt}$ $y' = f(y)$

$$\frac{u_{n+1} - u_n}{h} = f(u_n)$$ : finite difference scheme

$$\boxed{u_{n+1} = u_n + h\, f(u_n), \quad u_0 = y_0}$$

Ex $y' = y$, $y(0) = 1$ The exact solution is $y(t) = e^t$

In this case $f(y) = y$.

Suppose we want to compute $y(1) = 2.7182818$

Choose any $n \geq 1$, then $h = \frac{1-0}{n} = 1/n$.

$u_0 = 1$ since $y(0) = 1$

$u_1 = u_0 + h f(u_0) = u_0 + h u_0 = u_0(1+h) = 1+h$

$u_2 = u_1 + h f(u_1) = u_1 + h u_1 = u_1(1+h) = (1+h)^2$

. . . . .

$u_n = (1+h)^n$

| $n$ | $h$ | $u_n$ | global error $y(1) - u_n$ | $(y(1) - u_n)/h$ |
|---|---|---|---|---|
| 10 | 0.1 | 2.5937425 | 0.1245 | 1.245 |
| 20 | 0.05 | 2.6532977 | 0.0650 | 1.300 |
| 40 | 0.025 | 2.6850638 | 0.0332 | 1.328 |
| 80 | 0.0125 | 2.7014849 | 0.0168 | 1.344 |

Note. The table indicates that

1. $\lim_{n\to\infty} u_n = y(1)$, so that the method converges.

2. $u_n = y(1) + O(h)$, i.e. the method is 1st order accurate

Code

```
h=0.25                          t0=0
y0 = 1
tfinal = 2
nmax = tfinal / h
t = 0: 0.01 : tfinal                    f(y)=y
y=exp(t)
u(1) = y0
for n=1: nmax
   u(n+1) = u(n) + h * f(u(n))
end
tn= 0: h: tfinal
plot(t, y, tn, u, '--', tn,u, 'o')
```

Convergence (for Euler's method, special case)

$y' = y, \quad y(0) = 1, \quad y(1) = e \qquad y(t) = e^t$

$h = \frac{1-0}{n} = \frac{1}{n}$

$u_0 = 1$

$u_{n+1} = u_n + h \underbrace{f(u_n)}_{u_n} = u_n + h u_n = (1+h) u_n$

$\Rightarrow u_n = (1+h)^n$

$\lim_{n\to\infty} u_n \overset{?}{=} y(1) = e$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1+h)^n = \lim_{n\to\infty} (1+\frac{1}{n})^n =$

$= \lim_{n\to\infty} e^{\ln(1+\frac{1}{n})^n} = e^{\lim_{n\to\infty} \ln(1+\frac{1}{n})^n}$ $=1$

$\lim_{n\to\infty} \ln(1+\frac{1}{n})^n = \lim_{n\to\infty} n \cdot \ln(1+\frac{1}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \frac{0}{0}$

$n = \frac{1}{\frac{1}{n}}$

Use L'Hopital's rule

$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

Hence, $\lim_{n\to\infty} u_n = e^1 = e = y(1)$ $\underline{\underline{ok}}$

The local truncation error (local discretization error) is the amount by which the exact solution fails to satisfy the difference scheme

Ex

$y' = f(y)$, $y(0) = y_0 \Rightarrow y(t)$: exact solution

$u_{n+1} = u_n + h f(u_n)$, $u_0 = y_0$ : Euler's method

Let $y_n = y(t_n)$, where $t = t_n = nh$

$y_{n+1} = y_n + h f(y_n) + r_n$
where $r_n$ is local truncation error

Taylor expansion

$$y_{n+1} = y(t_{n+1}) = y(t_n + h) \underset{\text{Thm}}{\overset{\text{Taylor}}{=}} \text{ about } t = t_n$$

$$= y(t_n) + y'(t_n) \cdot h + \frac{h^2}{2} y''(\xi)$$

where $\xi$ is between $t_n$ and $t_n + h$

$$y_{n+1} = \underline{y_n} + h \cdot \underline{\underline{f(y_n)}} + r_n =$$

$$= \underline{y(t_n)} + \underline{\underline{y'(t_n)}} \cdot h + \frac{h^2}{2} y''(\xi)$$

$f(y_n) = y'(t_n)$ since $y' = f(y)$

Therefore, $r_n = \frac{h^2}{2} y''(\xi)$

If $y''(\xi)$ is bounded, i.e. there is $M > 0$:

$|y''(\xi)| \le M$, then

$$|r_n| \le \frac{h^2}{2} \cdot M = \underbrace{\frac{M}{2}}_{C} h^2 = C h^2 = O(h^2)$$

Conclusion

The local truncation error for Euler's method is $O(h^2)$.