Recall

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

Ex

$$f(x) = \frac{1}{1+25x^2} = \frac{1}{1-(5ix)^2} = \frac{1}{(1-5ix)(1+5ix)} =$$

use partial fraction decomposition

$\frac{1}{x}$

$$= \frac{1}{2}\left(\frac{1}{1-5ix} + \frac{1}{1+5ix}\right)$$

$n=1$ $-\frac{1}{x^2}$

$n=2$ $\frac{2}{x^3}$

$-\frac{2\cdot 3}{x^4}$

$$\boxed{f^{(n+1)}(x) = \frac{1}{2}(-1)^{n+1}(n+1)!\left(\frac{(-5i)^{n+1}}{(1-5ix)^{n+2}} + \frac{(5i)^{n+1}}{(1+5ix)^{n+2}}\right)}$$

$\frac{1}{1-5ix}$ $n=1$ $-\frac{1}{(1-5ix)^2}(-5i)$

$n=2$ $-\frac{1\,(-2)}{(1-5ix)^3}(-5i)^2 = (-1)^3\frac{(5i)^2\cdot 2}{(1-5ix)^3}$

If $\xi = 0$ $\left|\frac{f^{(n+1)}(0)}{(n+1)!}\right| \sim 5^{n+1}$

If $\xi = 1$ $\left|\frac{f^{(n+1)}(1)}{(n+1)!}\right| \sim \left(\frac{25}{26}\right)^{n+1}$

Uniform points: $x_i = -1 + \frac{2i}{n}$, $i = 0, \dots, n$

Chebyshev points: $x_i = -\cos\left(\frac{i\pi}{n}\right)$, $i = 0, \dots, n$

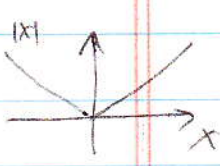
code (MATLAB)

```
n=4
x=-1+2*(0:n)/n
f=ones(x)./(1+25*x.*x)
a=polyfit(x, f, n)
x=-2:.01:2
f=ones(x)./(1+25*x.*x)
p=polyval(a,x)
plot(x,f, x,p)            plot(x,f, x,p,'--')
```

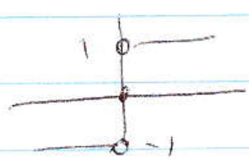
HW

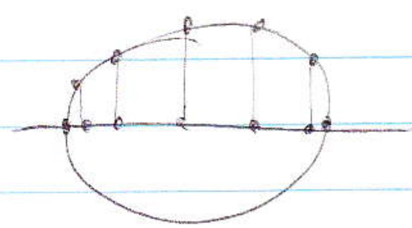
$n = 4, 8, 16$

$f_1(x) = |x| = abs(x)$

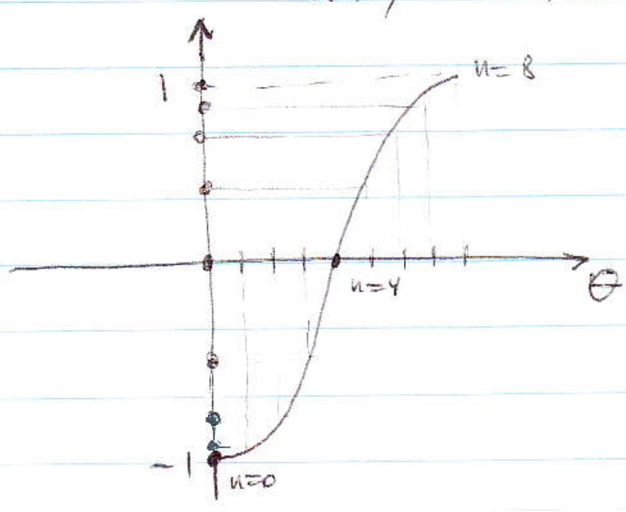


$$f_2(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases} = sign(x)$$





$x_i = -\cos\left(\frac{\pi i}{n}\right), \quad i = 0, 1, \dots, n$



n=4

$\cos\left(\frac{\pi}{2}\right)$

$\theta = \frac{\pi i}{n}$ $\qquad$ $\cos(\theta)$

n=8

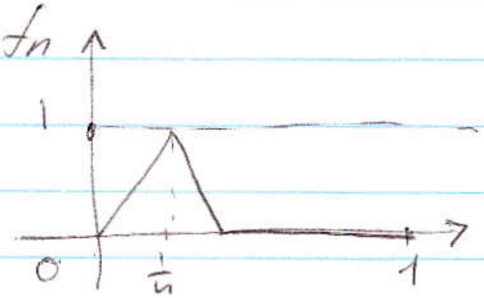
$i=0 \quad -\cos(0) = -1$

$i=1 \quad -\cos\left(\frac{\pi}{8}\right)$

$i=2 \quad -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

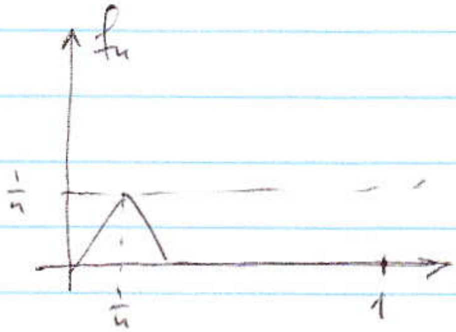
Pointwise / uniform convergence

Ex



$f_n \to 0$ pointwise on $[0,1]$

$f_n \not\to 0$ uniformly on $[0,1]$



$f_n \to 0$ pointwise on $[0,1]$

$f_n \to 0$ uniformly on $[0,1]$

1. $f_n(x) \to f(x)$ pointwise on $[0,1]$

$\iff \lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$

$\iff$

Given $x \in [0,1]$. For any $\varepsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N$$

$$N = N(x, \varepsilon)$$

$f_n \rightrightarrows f$

2. $f_n(x) \to f(x)$ uniformly on $[0,1]$

$\Leftrightarrow \lim_{n\to\infty} \max_{0\le x\le 1} \{|f_n(x) - f(x)|\} = 0$

$\Leftrightarrow$

For any $\varepsilon > 0$, there exists N such that

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$
and for all $n \geq N$

$N = N(\varepsilon)$