Note

If $f_n(x) \to f(x)$ uniformly on $[a, b]$, then $f_n(x) \to f(x)$ pointwise on $[a, b]$

$$f_n \rightrightarrows f(x) \quad \Rightarrow \quad f_n \to f$$
$$\not\Leftarrow$$

Note In general, If $f(x)$ and $f'(x)$ are continuous on $[-1, 1]$, then interpolating polynomial $p_n(x)$ based on Chebyshev points (Chebyshev interpolating polynomials) approach $f(x)$ uniformly on $[-1, 1]$.

Hermite interpolation

Problem

Given function $f(x)$, $x_0, x_1, .., x_n$: $n+1$ distinct points, find a polynomial p that interpolates f and f' at the given points, i.e.

$$p(x_i) = f(x_i) \qquad i = 0, 1, \dots, n$$
$$p'(x_i) = f'(x_i)$$

$p(x)$ is called Hermite interpolating polynomial, $\deg p \le 2n+1$

Define h_i, $\tilde{h}_i$ polynomials of degree $\le 2n+1$ such that

$$h_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad \tilde{h}_i(x_j) = 0$$

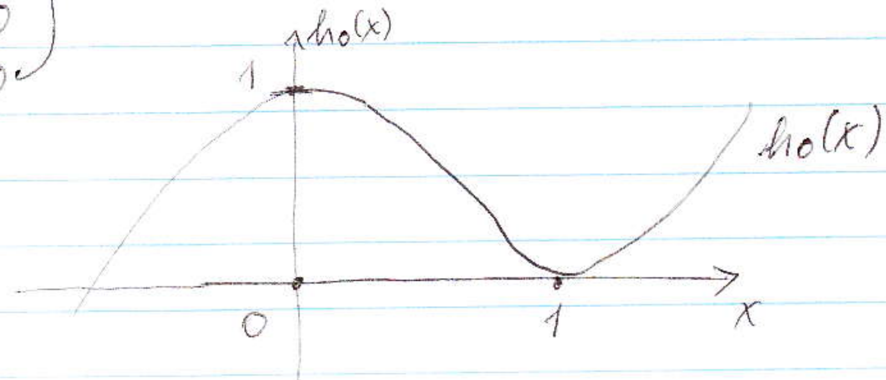
$$h_i'(x_j) = 0, \qquad \tilde{h}_i'(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Then
$$p(x) = \sum_{i=0}^{n} \left[f(x_i)\, h_i(x) + f'(x_i)\, \tilde{h}_i(x) \right]$$

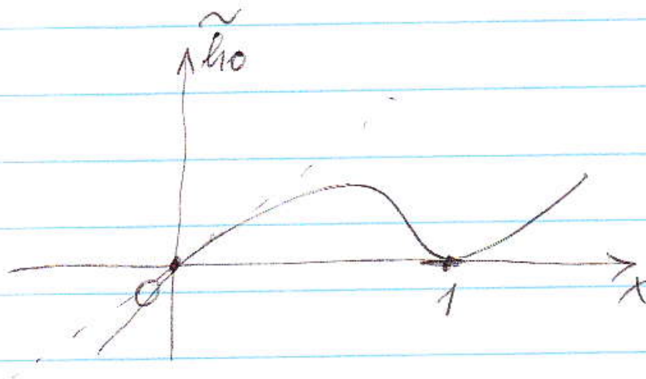
<u>Note</u>
This is similar to Lagrange interpolating polynomial.

<u>Ex</u> $n=1$ $\quad x_0 = 0, \quad x_1 = 1 \quad 2n+1 = 3$

$$\left.\begin{array}{l} h_0(0) = 1 \\ h_0(1) = 0 \\ h_0'(0) = 0 \\ h_0'(1) = 0 \end{array}\right\} \Rightarrow h_0(x) = (x-1)^2(2x+1)$$



$$\left.\begin{aligned} \tilde{h}_0(0)&=0 \\ \tilde{h}_0(1)&=0 \\ \tilde{h}_0'(0)&=1 \\ \tilde{h}_0'(1)&=0 \end{aligned}\right\} \Rightarrow \tilde{h}_0(x) = x(x-1)^2$$



Clearly, we need a more systematic way to construct h_i and $\tilde{h}_i$

<u>Claim</u> Let $l_i(x)$ be Lagrange polynomials,

$$l_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Then

$$h_j(x) = \left(1 - 2(x - x_j)\, l_j'(x_j)\right) l_j^2(x)$$

$$\tilde{h}_j(x) = (x - x_j)\, l_j^2(x)$$

<u>Check</u>

$$h_j(x_i) = \left(1 - 2(x_i - x_j)\, l_j'(x_j)\right) l_j^2(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$h_j'(x) = -2 l_j'(x_j) \cdot l_j^2(x) + \left(1 - 2(x - x_j)\, l_j'(x_j)\right) 2 l_j(x) * l_j'(x)$$

$$h_j'(x_i) = -2 l_j'(x_j)\, l_j^2(x_i) + \left(1 - 2(x_i - x_j)\, l_j'(x_j)\right) * * 2 l_j(x_i)\, l_j'(x_i) = \begin{cases} 0 & i = j \\ 0, & i \neq j \end{cases}$$

$$\tilde{h}_j(x_i) = (x_i - x_j)\, l_j^2(x_i) = \begin{cases} 0 & i=j \\ 0 & i\neq j \end{cases}$$

$$\tilde{h}_j'(x) = l_j^2(x) + (x - x_j)\, 2 l_j(x)\, l_j'(x)$$

$$\tilde{h}_j'(x_i) = l_j^2(x_i) + (x_i - x_j)\, 2 l_j(x_i)\, l_j'(x_i) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Newton's form $\{x_0, x_1\}$, $f(x)$

$$\begin{array}{llllll}
z_0 & x_0 & f[x_0] & & & \\
 & & & f'(x_0) & & \\
z_1 & x_0 & f[x_0] & & f[z_0, z_1, z_2] & \\
 & & & f[x_0, x_1] & & f[z_0, z_1, z_2, z_3] \\
z_2 & x_1 & f[x_1] & & f[z_1, z_2, z_3] & \\
 & & & f'(x_1) & & \\
z_3 & x_1 & f[x_1] & & &
\end{array}$$

$f[x_0] = f(x_0)$

$$p(x) = f[x_0] + f'(x_0)(x - z_0) + f[z_0, z_1, z_2]\overbrace{(x - z_0)(x - z_1)}^{(x-x_0)^2} +$$

$$+ f[z_0, z_1, z_2, z_3]\underbrace{(x - z_0)(x - z_1)(x - z_2)}_{(x-x_0)^2(x-x_1)}$$

$\underline{Ex}$ $x_0 = 0$, $f_0 = 1$, $f_0' = 0$

$x_1 = 1$, $f_1 = 0$, $f_1' = 0$

$$
\begin{array}{llllll}
z_0 & 0 & 1 & & & \\
 & & & 0 & & \\
z_1 & 0 & 1 & & \frac{-1-0}{1-0} = -1 & \\
 & & & \frac{0-1}{1-0} = -1 & & \frac{1-(-1)}{1-0} = 2 \\
z_2 & 1 & 0 & & \frac{0-(-1)}{1-0} = 1 & \\
 & & & 0 & & \\
z_3 & 1 & 0 & & &
\end{array}
$$

$$p(x) = 1 + 0\cdot(x-0) + (-1)(x-0)^2 + 2(x-0)^2(x-1) =$$

$$= 1 - x^2 + 2x^2(x-1) = (1-x)(1+x) + 2x^2(x-1) =$$

$$= (x-1)(-1-x+2x^2) = (x-1)(2x^2-x-1) = (x-1)^2(2x+1) \underset{}{=} h_0(x) \checkmark$$

$$p(x) = \underset{1}{\underset{\|}{f_0}} \cdot h_0(x) + \cancelto{0}{f_1}\, h_1(x) + \cancelto{0}{f_0'}\, \tilde{h}_0(x) + \cancelto{0}{f_1'}\, \hat{h}_1(x) = h_0(x)$$

$$
\begin{array}{lll}
x_0 & f[x_0] & \\
 & & \dfrac{f[x_0+\varepsilon] - f[x_0]}{x_0+\varepsilon-x_0} \to f'(x_0) \\
x_0+\varepsilon & f[x_0+\varepsilon] &
\end{array}
$$

$$
\begin{array}{llll}
x_0 & f[x_0] & & \\
 & & f'(x_0) & \\
x_0 & f[x_0] & & \frac{1}{2!} f''(x_0) \\
 & & f'(x_0) & \\
x_0 & f[x_0] & &
\end{array}
$$

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

$$p_n(x) = \sum_{j=0}^{n} f(x_j)\cdot l_j(x)$$

$$= f(x_0)l_0(x) + f(x_1)l_1(x) + \dots + f(x_n)l_n(x)$$

$$\underset{1}{\overset{\shortparallel}{f(x)}} = \underbrace{\underbrace{f(x_0)}_{1} l_0(x) + \dots + \underbrace{f(x_n)}_{1} l_n(x)}_{\sum_{j=0}^{n} l_j(x)} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\dots$$

$f \equiv 1$