4/19/2010

Piecewise polynomial interpolation

Given function f, $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$

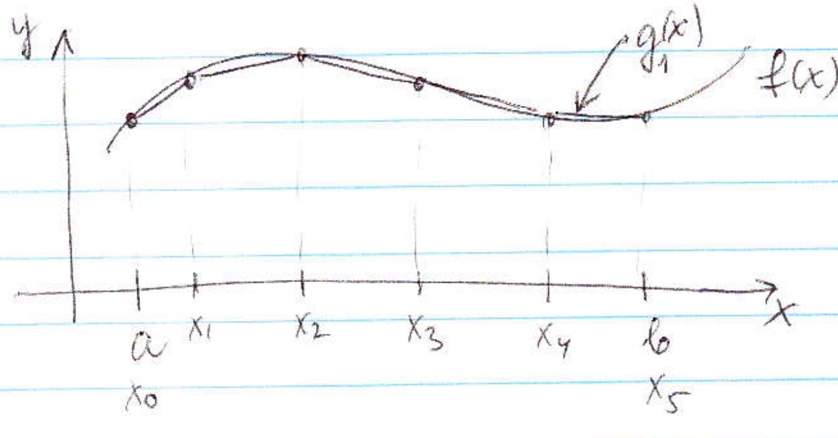
The interpolating polynomial of degree $\leq n$, p_n, may not give a good approximation over the entire interval $[a, b]$.

Define g_1: piecewise linear interpolant by

$$g_1(x) = f[x_i] + f[x_i, x_{i+1}](x - x_i), \quad x \in [x_i, x_{i+1}]$$

$$g_1(x_i) = f[x_i] = f(x_i)$$

$$g_1(x_{i+1}) = f[x_i] + \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}(x_{i+1} - x_i) = f(x_{i+1})$$



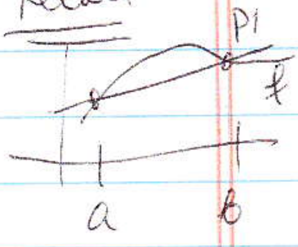
g_1 is continuous on $[a, b]$ but not differentiable on $[a, b]$ (g_1 is not differentiable at x_i, $i = 1, 2, .., n-1$

Let $M_i = \max\{|f''(x)|, \ x \in [x_i; x_{i+1}]\}$

For any $x \in [a, b]$,

$$|f(x) - g_1(x)| \leq \max_i \left\{ \frac{M_i}{8} |x_{i+1} - x_i|^2 \right\}$$

Recall

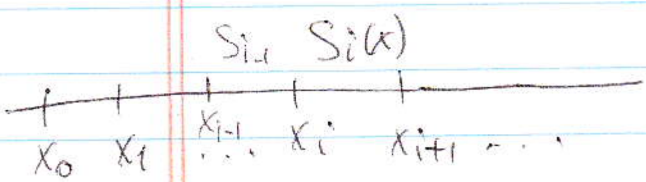


$$\|f - p_1\|_\infty \leq \frac{M}{8}(b-a)^2, \text{ where } M = \max_{[a,b]}(f'')$$

Splines

Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.
A spline of degree m is a function $S(x)$ that satisfies the following conditions:

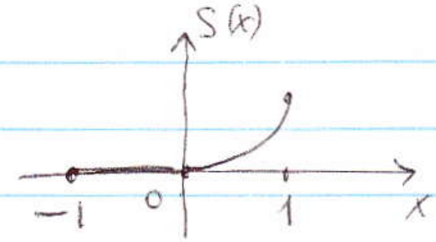
1. For $x \in [x_i, x_{i+1}]$, $S(x) = S_i(x)$ a polynomial of degree $\leq m$



2. $S^{(m-1)}(x)$ exists and continuous at interior points $x_1, \ldots, x_{n-1}$, i.e.

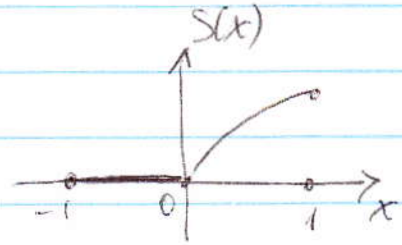
$$\lim_{x \to x_i^-} S_{i-1}^{(m-1)}(x) = \lim_{x \to x_i^+} S_i^{(m-1)}(x)$$

Ex $x_0 = -1,\ x_1 = 0,\ x_2 = 1$



$$S(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1 \end{cases}$$

Yes, $S(x)$ is a spline of degree $2 = m$ (quadratic spline).



Ex $S(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1-(x-1)^2, & 0 \le x \le 1 \end{cases}$

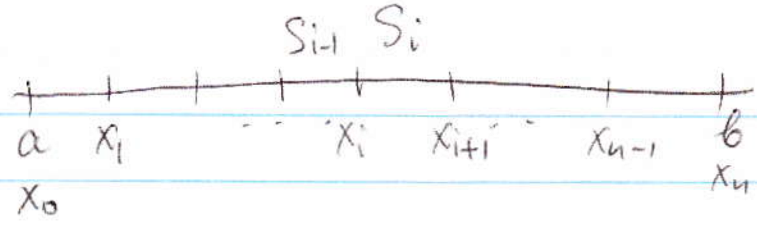
$$\lim_{x\to 0+} S'(x) = \lim_{x \to 0+}\left(-2(x-1)\right) = 2 \ne \lim_{x\to 0-} S'(x) = 0$$

No, $S(x)$ is NOT a quadratic spline since $\lim_{x\to 0-} S'(x) \ne \lim_{x \to 0+} S'(x)$

Cubic spline interpolation

Given f, $x_0, x_1, \dots, x_n$ as above, find a cubic spline $S(x)$ that interpolates function $f(x)$:
$f(x_i) = S(x_i)$, $i = 0, 1, \dots, x_n$.

$n+1$ points $\Rightarrow$ n intervals $\Rightarrow$ $4n$ coefficients



$2n = 2(n-1) + 2$ conditions to interpolate f

$2(n-1)$ conditions to require that $S'(x)$ and $S''(x)$ are continuous at interior points $x_1, \dots, x_{n-1}$

$\Rightarrow$ we have $4n-2$ conditions and $4n$ unknowns

$\Rightarrow$ we need two extra conditions

A popular choice is

$S''(x_0) = S''(x_n) = 0$: natural cubic spline interpolant

Another choice:

$S'(x_0) = f'(x_0)$, $\quad S'(x_n) = f'(x_n)$: clamped cubic spline interpolant