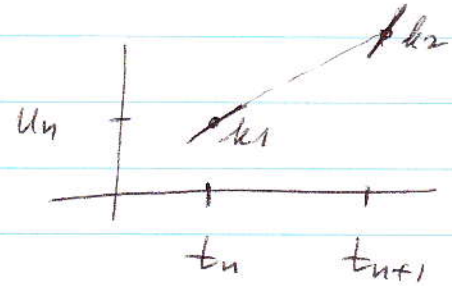
Modified Euler Method

$$y' = f(y), \quad y(0) = y_0$$

Runge-Kutta approach

$$\begin{cases} k_1 = f(u_n) \\ k_2 = f(u_n + h k_1) \\ u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) \end{cases}$$



predictor - corrector approach

$$\begin{cases} \tilde{u}_{n+1} = u_n + h \cdot f(u_n) & \text{predictor (Euler)} \\ u_{n+1} = u_n + \frac{h}{2}\left(f(u_n) + f(\tilde{u}_{n+1})\right) & \text{corrector} \end{cases}$$

Note

1. This is an example of 2 stage Runge-Kutta method
2. Each step of modified Euler method requires two function evaluations.

Ex $\quad y' = y, \quad y(0) = 1, \quad y(1) = 2.7182818\ldots$

$f(y) = y$

$$k_1 = f(u_n) = u_n$$

$$k_2 = f(u_n + h k_1) = u_n + h k_1 = $$
$$= u_n + h u_n = (1+h) u_n$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) = u_n + \frac{h}{2}\left(u_n + (1+h)u_n\right)$$

$$= u_n + u_n \cdot \frac{h}{2}(2+h) = u_n\left(1 + h + \frac{h^2}{2}\right)$$

$$u_{n+1} = u_n\left(1+h+\frac{h^2}{2}\right), \quad h=\frac{1}{n}$$

n	h	u_n	$y(1)-u_n$	$(y(1)-u_n)/h^2$
10	0.1	2.71408085	0.00420098	0.4201
20	0.05	2.71719105	0.00109077	0.4363
40	0.025	2.71800394	0.00027788	0.4446
80	0.0125	2.71821170	0.00007013	0.4488

Note

The table indicates that

1. Modified Euler's method converges
2. Global error is of 2^{nd} order, ie.

$$u_n = y(1) + O(h^2)$$

ie. modified Euler's method is 2^{nd} order accurate

Claim

The local truncation error for modified Euler's method is $O(h^3)$.

Note

The significance of the local truncation error is that if $\tau_n = O(h^{p+1})$, then the global error (error at the final time or cumulative error) is $O(h^p)$, ie. $|u_n - y_n| = O(h^p)$

4th order Runge–Kutta method

$$y' = f(y)$$

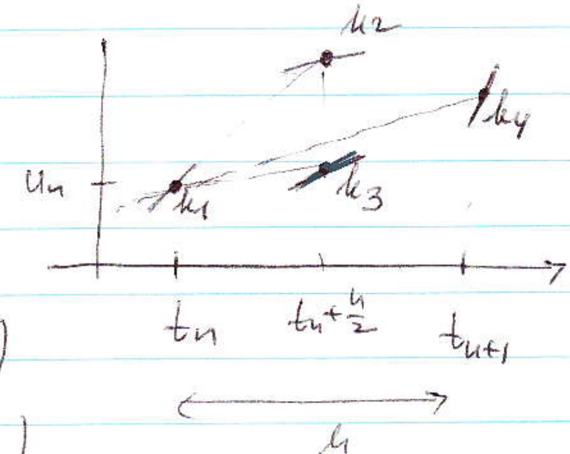
$$k_1 = f(u_n)$$

$$k_2 = f\left(u_n + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(u_n + \frac{h}{2} k_2\right)$$

$$k_4 = f\left(u_n + h \cdot k_3\right)$$

$$u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$



$$y' = f(t, y)$$

$$k_1 = f(t_n, u_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} k_2\right)$$

$$k_4 = f\left(t_n + h, u_n + h \cdot k_3\right)$$

$$u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Note

1. The method requires 4 function evaluations per time step.
2. The local truncation error is $\tau_n = O(h^5)$
3. The method is 4th order accurate.

Ex $y' = y, \quad y(0) = 1, \quad y(1) = 2.7182818\ldots$

$f(y) = y$

$$k_1 = f(u_n) = \boxed{u_n}$$

$$k_2 = f(u_n + \tfrac{h}{2} k_1) = u_n + \tfrac{h}{2} k_1 = u_n + \tfrac{h}{2} u_n = \boxed{u_n(1 + \tfrac{h}{2})}$$

$$k_3 = f(u_n + \tfrac{h}{2} k_2) = u_n + \tfrac{h}{2} k_2 = u_n + \tfrac{h}{2}\left(1 + \tfrac{h}{2}\right) u_n =$$

$$= \boxed{u_n\left(1 + \frac{h}{2} + \frac{h^2}{4}\right)}$$

$$k_4 = f(u_n + h \cdot k_3) = u_n + h \cdot k_3 = u_n + h \cdot \left(1 + \frac{h}{2} + \frac{h^2}{4}\right) u_n$$

$$= \boxed{u_n\left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right)}$$

$$u_{n+1} = u_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) =$$

$$= u_n + \frac{h}{6}\left(u_n + 2u_n\left(1 + \frac{h}{2}\right) + 2u_n\left(1 + \frac{h}{2} + \frac{h^2}{4}\right) + \right.$$

$$\left. + u_n\left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right)\right) = \ldots \text{ algebra gives} \ldots$$

$$u_{n+1} = \left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) u_n$$

<u>Note</u> solution at t_{n+1} depends only on solution at t_n $\Rightarrow$ R-K is one-step method but 4 stage method.

n	h	u_n	$y(1)-u_n$	$(y(1)-u_n)/h^4$
1	1.0	2.70833333	0.00994850	0.0099
2	0.5	2.71734619	0.00095637	0.0150
4	0.25	2.71820944	0.00007189	0.0184
8	0.125	2.71827684	0.00000498	0.0204

Test Equation

$y' = \lambda y, \quad \lambda \in \mathbb{C}$

$\frac{dy}{dt} = \lambda y$

$\frac{dy}{y} = \lambda dt$

$\ln|y| = \lambda t + C_1$

$y = C_2 e^{\lambda t}$

IC $y(0) = y_0 \Rightarrow C_2 = y_0 \Rightarrow$ solution is

$$\boxed{y(t) = y_0 e^{\lambda t}}$$

$$y_{n+1} = y(t_{n+1}) = y(t_n + h) = y_0 e^{\lambda(t_n+h)} = \underbrace{y_0 e^{\lambda t_n}}_{y(t_n)} \cdot e^{\lambda h}$$

$$\Rightarrow y_{n+1} = e^{\lambda h} y_n$$

The <u>amplification factor</u> of the exact equation, $y' = \lambda y$ is $e^{\lambda h}$.

Euler's method: $u_{n+1} = u_n + h \cdot \underbrace{f(u_n)}_{\lambda u_n} = u_n + h \cdot \lambda u_n$

$y' = \underbrace{\lambda y}_{f(y)}$

$$u_{n+1} = (1 + h\lambda) u_n$$

The amplification factor of Euler's equation is $1 + h\lambda$

For various numerical method, the amplification factor is some function of $h\lambda$, i.e.

$$u_{n+1} = Q(h\lambda) u_n$$

<u>Claim</u>

The numerical method is of order p $\Leftrightarrow$ $Q(h\lambda) = e^{h\lambda} + O(h^{p+1})$

<u>Ex</u> Euler's method

$Q(h\lambda) = 1 + h\lambda$

$e^{h\lambda} = 1 + h\lambda + \frac{h^2\lambda^2}{2!} + O(h^3) \Rightarrow 1 + h\lambda = e^{h\lambda} + O(h^2)$

$\Rightarrow$ Euler's method is 1^{st} order accurate