Systems

$$y_1' = f_1(y_1, \dots, y_n) \qquad ' = \frac{d}{dt}$$

$$\dots\dots$$

$$y_n' = f_n(y_1, \dots, y_n)$$

Vector form

$$y' = f(y) \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Special case

$$y_1' = a \cdot y_1 + b \cdot y_2$$

$$y_2' = c y_1 + d \cdot y_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y' = Ay \qquad f(y) = Ay$$

Euler's method

$$u_{n+1} = u_n + h f(u_n)$$

$$u_{n+1} = u_n + h \cdot A u_n = (I + hA) u_n$$

Modified Euler's method

$f(y) = Ay$

$$k_1 = Au_n \qquad\qquad k_1 = f(u_n)$$

$$k_2 = f(u_n + h k_1)$$

$$k_2 = A(u_n + h k_1) = \qquad\qquad u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2)$$

$$= A(u_n + hAu_n)$$

$$= A(I + hA)u_n = (A + hA^2)u_n$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) = u_n + \frac{h}{2}\Big(Au_n +$$

$$+ (A + hA^2)u_n\Big) = \left(I + \frac{h}{2}A + \frac{h}{2}A + \frac{h^2}{2}A^2\right)u_n$$

$$u_{n+1} = \left(I + hA + \frac{h^2}{2}A^2\right)u_n$$

Exact solution

$$y' = Ay, \quad y(0) = y_0 \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Assume that matrix A can be diagonalized
with e'values $\lambda_1, \lambda_2 \in \mathbb{C}$
e'vectors p_1, p_2

$$Ap_1 = \lambda_1 p_1$$

$$Ap_2 = \lambda_2 p_2$$

If $\lambda_1 \neq \lambda_2$, then e'vectors p_1, p_2 are linearly independent and can form a basis in solution space.

Let $y(t)$ be a solution of $y' = Ay$, then

$y(t) = \alpha_1(t) p_1 + \alpha_2(t) p_2$

at $t=0$

$$y(0) = \alpha_1(0) p_1 + \alpha_2(0) p_2 = y_0$$

This determines $\alpha_1(0)$ and $\alpha_2(0)$.

$$y' = Ay \iff \alpha_1' p_1 + \alpha_2' p_2 = A(\alpha_1 p_1 + \alpha_2 p_2)$$

$$\alpha_1' p_1 + \alpha_2' p_2 = \alpha_1 \underbrace{A p_1}_{=\lambda_1 p_1} + \alpha_2 \underbrace{A p_2}_{=\lambda_2 p_2}$$

$$\alpha_1' p_1 + \alpha_2' p_2 = \alpha_1 \lambda_1 p_1 + \alpha_2 \lambda_2 p_2$$

p_1, p_2 are linearly independent

$\Rightarrow \alpha_1' = \alpha_1 \lambda_1, \quad \alpha_2' = \alpha_2 \lambda_2$: scalar equations

$\Rightarrow \boxed{y(t) = \alpha_1(0) e^{\lambda_1 t} \cdot p_1 + \alpha_2(0) e^{\lambda_2 t} p_2}$ exact solution

<u>Numerical solution</u>

<u>Ex</u> Euler's method

$$y' = Ay = f(y)$$

$$u_{n+1} = u_n + h f(u_n) = u_n + h A u_n = (I + hA) u_n$$

$$u_0 = y_0 = \alpha_1(0) p_1 + \alpha_2(0) p_2$$

$$u_1 = (I + hA) u_0 = (I + hA)\left(\alpha_1(0) p_1 + \alpha_2(0) p_2\right) =$$

$$= \alpha_1(0) p_1 + \alpha_2(0) p_2 + h\alpha_1(0) \underbrace{A p_1}_{\lambda_1 p_1} + h\alpha_2(0) \underbrace{A p_2}_{\lambda_2 p_2}$$

$$= \alpha_1(0)(1 + h\lambda_1) p_1 + \alpha_2(0)(1 + h\lambda_2) p_2$$

$$u_2 = \alpha_1(0)(1 + h\lambda_1)^2 p_1 + \alpha_2(0)(1 + h\lambda_2)^2 p_2$$

$$\text{- - - - - - -}$$

$$\boxed{u_n = \alpha_1(0)(1 + h\lambda_1)^n p_1 + \alpha_2(0)(1 + h\lambda_2)^n p_2}$$

numerical solution

$$\boxed{y(t) = \alpha_1(0) e^{\lambda_1 t} p_1 + \alpha_2(0) e^{\lambda_2 t} p_2}$$

exact solution

Note

The exact solution is bounded for all $t > 0$ and all $y_0 \iff$

$$Re(\lambda_1) \leq 0 \quad \text{and} \quad Re(\lambda_2) \leq 0$$

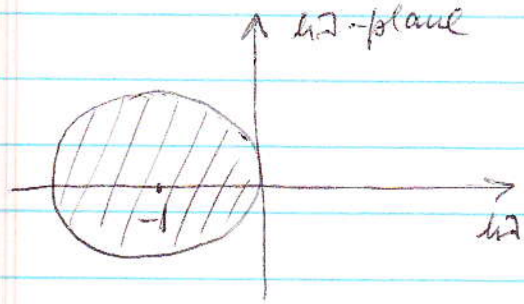
Numerical solution is bounded for all $n > 0$ and all $y_0 \iff$

$|1+h\lambda_1| \le 1$ and $|1+h\lambda_2| \le 1$

the region in $h\lambda$-plane which satisfies these conditions is called <u>the region of absolute stability</u> for Euler's method.

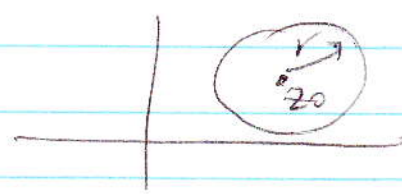
$|1+h\lambda| \le 1$

$|1+z| \le 1$



$|z+1| \le 1$

$|z-z_0| \le r$

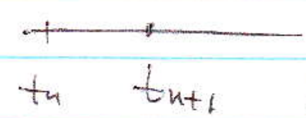


$|1+h\lambda| \le 1 \Leftrightarrow |h\lambda-(-1)| \le 1$

<u>Backward Euler method</u>

$y' = f(y)$

$$\frac{u_{n+1}-u_n}{h} = f(u_{n+1})$$



Forward Euler

$$\frac{u_{n+1}-u_n}{h} = f(u_n)$$

$u_{n+1} = u_n + h f(u_{n+1})$

This method is <u>implicit</u> (Forward Euler is <u>explicit</u>)