5/7/2010

$$E = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1)\dots(x-x_n)}_{\omega(x)}$$

$\omega_{10}(x)$

Ex $f = \frac{1}{1+25x^2}$ $\qquad f = p_n(x) + E_n \not\to 0$ as $n \to \infty$

Runge phenomenon

$Ax = b \not\Rightarrow x = A^{-1}b$

Claim: Backward Euler's method is 1st order accurate

Pf $\qquad u_{n+1} = u_n + h\, f(u_{n+1})$

$y' = f(y)$

$y_{n+1} = y_n + h f(y_{n+1}) + r_n$ local truncation error

$y_{n+1} = y(t_{n+1}) = y(t_n + h) \overset{\text{Taylor}}{=} y(t_n) + \underbrace{y'(t_n)}_{f(y_n)} \cdot h + O(h^2)$

$\Rightarrow y_n + h f(y_{n+1}) + r_n = y_n + h \cdot f(y_n) + O(h^2)$

$f(y_{n+1}) = y'_{n+1} = y'(t_{n+1}) = y'(t_n + h) = y'(t_n) + y''(t_n) h + O(h^2)$

$= y'(t_n) + O(h)$

~~$y_n + h(t_n)$~~

$$\underline{y_n} + h\left(\underbrace{y'(t_n)}_{\underline{f(y_n)}} + O(h)\right) + r_n = \underline{y_n} + \underline{h \cdot f(y_n)} + O(h^2)$$

$$\Rightarrow \boxed{r_n = O(h^2)} \Rightarrow \text{1st order method}$$

$$h \cdot O(h) + r_n = O(h^2)$$

<u>absolute stability</u> $\qquad y' = \lambda y$

$y' = \lambda y$

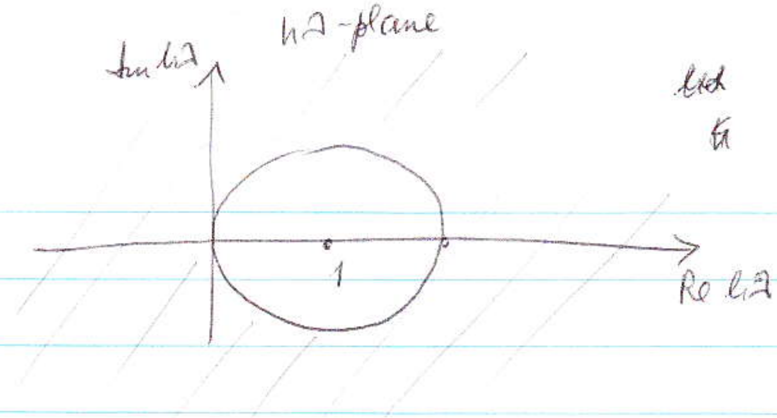
$$u_{n+1} = u_n + h f(u_{n+1}) = u_n + h \cdot \lambda u_{n+1}$$

$$(1 - h\lambda) u_{n+1} = u_n \Rightarrow u_{n+1} = \underbrace{\frac{1}{1-h\lambda}}_{\text{amplification factor}} u_n$$

$$\Rightarrow u_n = \underbrace{\left(\frac{1}{1-h\lambda}\right)}_{\text{aw}}^n u_0$$

u_n is bounded for all n $\Rightarrow \left|\frac{1}{1-h\lambda}\right| \leq 1$

or $\quad |h\lambda - 1| \geq 1$



no restriction on h since $\mathrm{Re}\,\lambda \leq 0$.

Method is called ~~A~~ A-stable because the region of absolute stability includes the entire left half plane.

Def
A system of ODEs, $y' = Ay$, is called a stiff system if A has negative e'values with greatly different magnitudes.

Ex

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Assume $\lambda_1 \ll \lambda_2 < 0$

$$y(t) = \alpha_1(0) e^{\lambda_1 t} p_1 + \alpha_2(0) e^{\lambda_2 t} p_2 : \text{exact}$$

$$u_n = \alpha_1(0)(1 + h\lambda_1)^n p_1 + \alpha_2(0)(1 + h\lambda_2)^n p_2 :$$

Euler method

$$u_n = \alpha_1(0)\left(\frac{1}{1 - h\lambda_1}\right)^n p_1 + \alpha_2(0)\left(\frac{1}{1 - h\lambda_2}\right)^n p_2 :$$

backward Euler

We require both $h\lambda_1$ and $h\lambda_2$ to be in the region of absolute stability. For Euler's method we impose:

$$h \le \frac{-1}{\lambda_1}$$

For ~~the~~ backward Euler, there is no restriction on h.

Note

Implicit schemes are often implemented as predictor - corrector methods

Ex $u_{n+1} = u_n + h f(u_{n+1})$

$$\Rightarrow \quad u_{n+1}^{(1)} = u_n + h f(u_n) : \text{predictor}$$

$$u_{n+1}^{(k+1)} = u_n + h f(u_{n+1}^{(k)}) : \text{corrector}$$

Review Romberg integration
or Richardson extrapolation technique

$$f'(x) = Dh + \text{error}$$

$$\text{error} = \cancel{Ch}\ O(h)$$

$$f'(x) = Df + C_1 h + C_2 h^2 + C_3 h^3 + \dots$$