Summary: Boundary Value Problems for $\frac{d^2\phi}{dx^2} = -\lambda\phi$

	Dirichlet	Neumann	Periodic
BCs	$\phi(0) = \phi(L) = 0$	$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$, $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
e'values	$\left(\frac{n\pi}{L}\right)^2$, $n=1,2,\ldots$	$\left(\frac{n\pi}{L}\right)^2$, $n=0,1,2,\ldots$	$\left(\frac{n\pi}{L}\right)^2$, $n=0,1,2,\ldots$
E'functions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$, $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, $n\geq 1$	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, $n\geq 1$

Steady-state heat equation in 2D

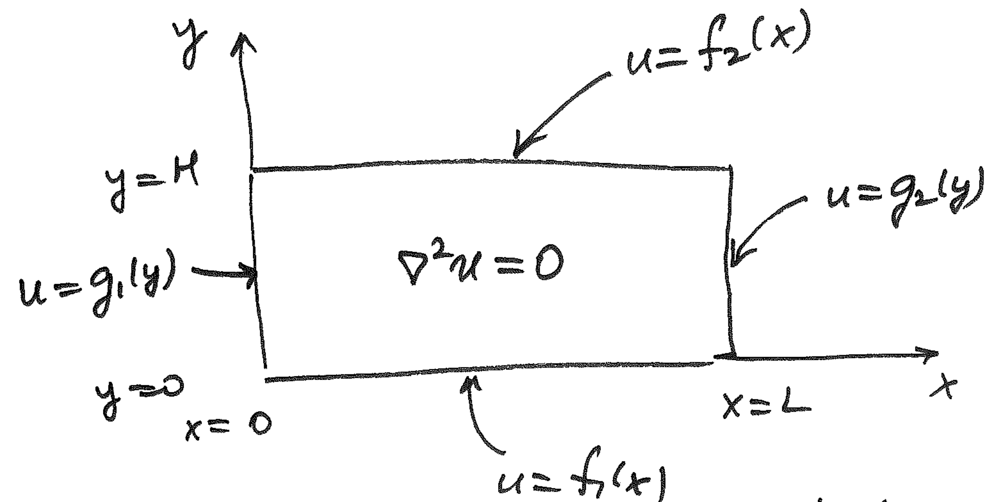
$\nabla^2 u = u_{xx} + u_{yy} = 0$, $\quad 0 \le x \le L$, $\; 0 \le y \le H$

$u(0,y) = g_1(y)$

$u(L,y) = g_2(y)$

$u(x,0) = f_1(x)$

$u(x,H) = f_2(x)$



Note: The PDE $\nabla^2 u = 0$ is linear and homogeneous but BCs are not homogeneous (If BCs were homogeneous, trivial solution $u \equiv 0$ would satisfy them). Hence, we cannot apply separation of variables directly.

Superposition: turn this problem into 4 subproblems.

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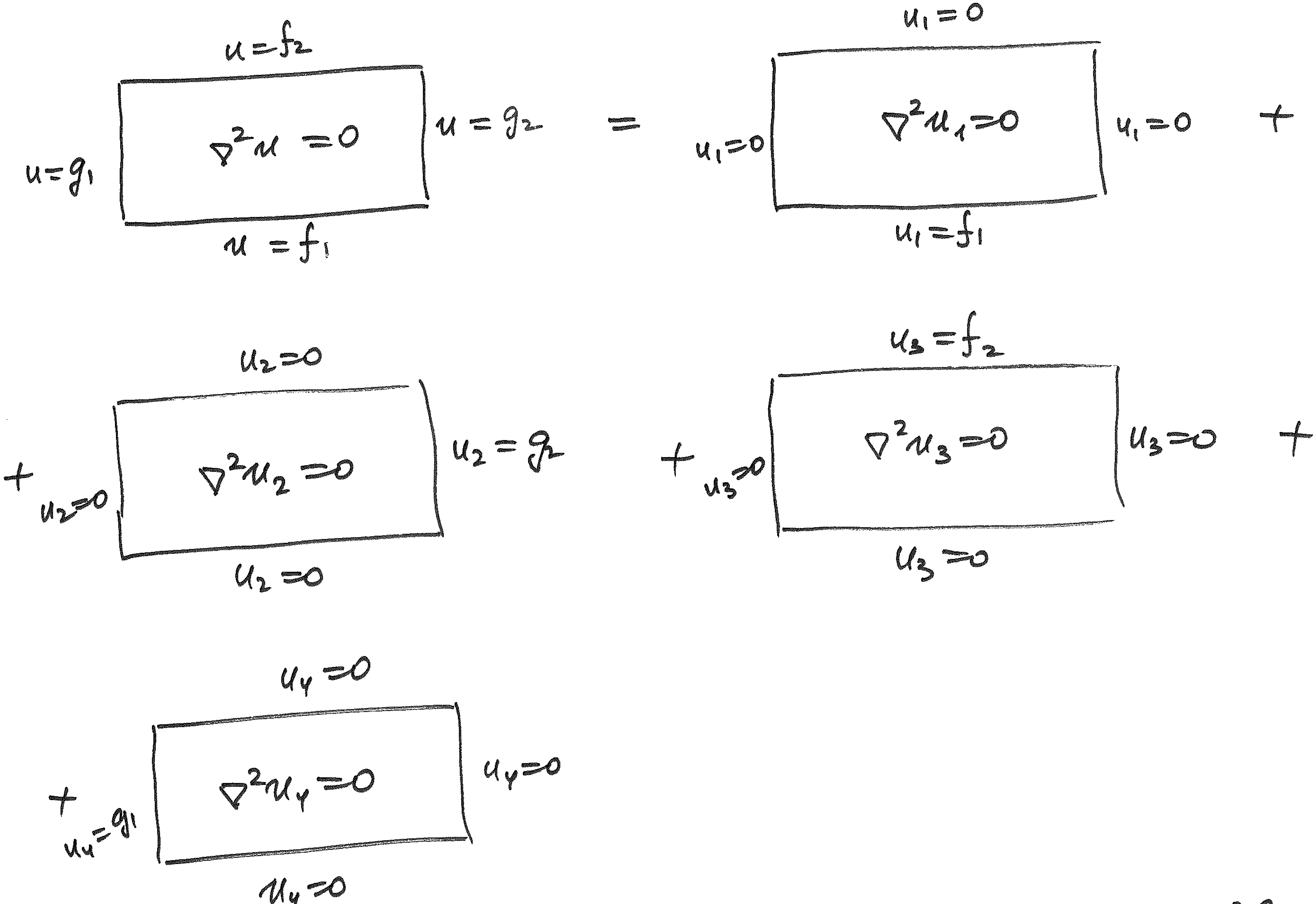

Then $u = u_1 + u_2 + u_3 + u_4$ satisfies the PDE and BCs.

Problem #4

$$\nabla^2 u_4 = \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0, \qquad 0 \le x \le L_1 \qquad 0 \le y \le H$$

$u_4(x,0) = 0$ $\qquad u_4(L_1,y) = 0$

$u_4(x,H) = 0$ $\qquad u_4(0,y) = g_1(y)$

Note: we have 3 homogeneous BCs and 1 nonhomogeneous.

Separation of variables: $u_4(x,y) = h(x)\,\phi(y)$

$$\frac{d^2 h}{dx^2}\phi(y) + h(x)\frac{d^2\phi}{dy^2} = 0 \quad \Big| \; \frac{1}{h(x)\,\phi(y)}$$

$$\underbrace{\frac{d^2 h/dx^2}{h(x)}}_{\text{function of } x} = \underbrace{-\frac{d^2\phi/dy^2}{\phi(y)}}_{\text{function of } y} = \lambda$$

BCs:

$u_y(x,0)=0 \Rightarrow h(x)\,\phi(0)=0 \Rightarrow \boxed{\phi(0)=0}$

$u_y(x,H)=0 \Rightarrow \boxed{\phi(H)=0}$

$u_y(L,y)=0 \Rightarrow h(L)\,\phi(y)=0 \Rightarrow \boxed{h(L)=0}$

Hence,

$$\frac{d^2h/dx^2}{h(x)} = -\frac{d^2\phi/dy^2}{\phi(y)} = \lambda$$

$\phi(0)=0$ $\phi(H)=0$

$h(L)=0$

<u>Note</u> We have 2 point BVPs for $\phi(y)$ and $h(x)$. Since BVP for $h(x)$ has only 1 BC, this problem is not strictly speaking a BVP.

$$\frac{d^2\phi}{dy^2} + \lambda\phi = 0 \qquad \phi(0)=0 \qquad \phi(H)=0$$

Since the BVP for ϕ has both zero BCs at endpoints, this suggests that solution for ϕ is oscillatory $\Rightarrow$ this gives information about the sign of λ $(\lambda > 0)$:

$$\left.\begin{aligned} &\frac{d^2\phi}{dy^2} + \lambda\phi = 0 \\ &\phi(0) = 0 \\ &\phi(H) = 0 \end{aligned}\right\}$$

We solved this e'value problem when we considered Dirichlet BCs.

Here $L = H$

e'values: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2, \ldots$

e'functions: $\phi_n(y) = \sin\frac{n\pi y}{L} = \sin\frac{n\pi y}{H}$

Problem for $h(x)$

$$\frac{d^2h}{dx^2} - \lambda h = 0$$

$$h(L) = 0$$

$\lambda = \lambda_n = \left(\frac{n\pi}{H}\right)^2 > 0 \Rightarrow$ roots of charact. eq^n are $\pm\sqrt{\lambda_n}$

$\therefore$ solutions are $e^{\pm\sqrt{\lambda_n}\,x}$ or $\cosh\sqrt{\lambda_n}\,x$ and $\sinh\sqrt{\lambda_n}\,x$

It is convenient (because of BC at $x=L$) to write solution in the form

$$h(x) = a_1 \cancel{\cosh} \frac{n\pi}{H}(x-L) + a_2 \sinh \frac{n\pi}{H}(x-L)$$

$\to 0$

at $x=L$:

$$\underset{\substack{\| \\ 0}}{h(L)} = \underbrace{a_1 \cosh \underbrace{\frac{n\pi}{H}(0)}_{0}}_{\cosh 0 = 1} + \underbrace{a_2 \sinh \frac{n\pi}{H}(0)}_{0} = a_1 + a_2 \cdot 0$$

$$\Rightarrow a_1 + a_2 \cdot 0 = 0 \quad \Rightarrow \boxed{a_1 = 0}$$

$$\therefore \; h(x) = a_2 \sinh \frac{n\pi}{H}(x-L)$$

Product solution: $u_4(x,y) = A \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x-L)$

Note: solution oscillates in y and does not oscillate in x.

This is a typical property of solutions of Laplace's equation.

General solution:

$$\boxed{u_4(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x-L)}$$

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To find A_n's we use BC at $x=0$ and orthogonality of sines.

$$u_y(0,y) = g_1(y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H} \underbrace{\sinh\left(\frac{n\pi}{H}(-L)\right)}_{= -\sinh\frac{n\pi L}{H}}$$

$$u_y(0,y) = g_1(y)$$

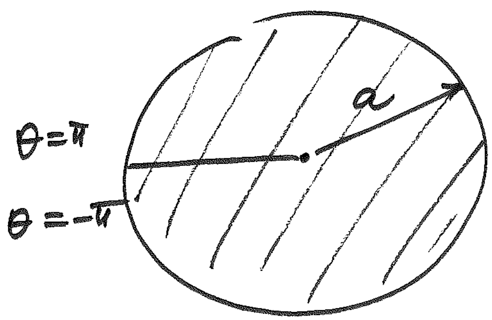
$$g_1(y) = \sum_{n=1}^{\infty} \underbrace{-A_n \sinh\frac{n\pi L}{H}}_{\text{Fourier coefficients}} \sin\frac{n\pi y}{H} \qquad \Big/ \cdot \sin\frac{m\pi y}{H} \quad \int_0^H$$

Use orthogonality of $\sin\frac{n\pi y}{H}$.

$$\int_0^H g_1(y)\sin\frac{m\pi y}{H}\,dy = -A_m \sinh\frac{m\pi L}{H} \underbrace{\int_0^H \sin^2\left(\frac{m\pi y}{H}\right) dy}_{=\frac{H}{2}}$$

$$\therefore \quad \boxed{A_m = -\frac{2}{H}\frac{1}{\sinh\frac{m\pi L}{H}} \int_0^H g_1(y)\sin\frac{m\pi y}{H}\,dy}$$

Laplace's Equation on a Circular Disk



$-\pi \leq \theta \leq \pi$ $\qquad$ $0 \leq r \leq a$

- Steady-state solution for heat distribution on a disk

- Assume that all thermal properties are const
- We have prescribed temperature at the boundary.