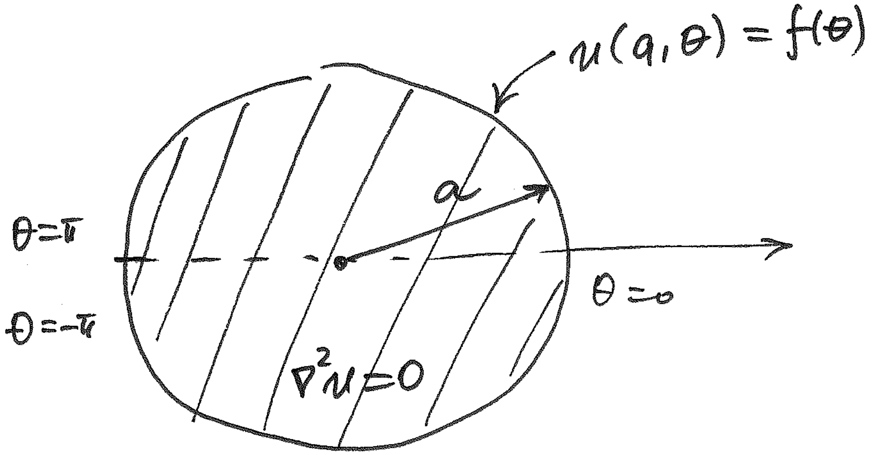
Lecture 11

Laplace's Equation on a Circular Disk



Laplace eq^n can be thought as steady-state version of heat eq^n

$$\cancel{u_t} \overset{0}{=} k \nabla^2 u \Rightarrow \nabla^2 u = 0$$

Because of geometry we will use polar coordinates (r, θ)

$$-\pi \le \theta \le \pi, \quad 0 \le r \le a$$

a: radius of the disk

$u(r,\theta)$: temperature

$u(r,\theta)\big|_{r=a} = f(\theta)$: prescribed temperature at the boundary

We assume constant thermal properties

In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$u(a,\theta) = f(\theta)$: prescribed temperature ~~at~~ $r=a$

$|u(0,\theta)| < \infty$: boundedness at the origin

$$\left.\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ u_\theta(r,-\pi) &= u_\theta(r,\pi) \end{aligned}\right\} \text{ periodic BCs}$$

Separation of Variables

$$u(r,\theta) = G(r)\,\phi(\theta)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2}\cdot G(r) = 0 \quad \Bigg| \quad \frac{1}{\frac{1}{r^2}G(r)\phi(\theta)}$$

$$\frac{r\frac{d}{dr}\left(r\frac{dG}{dr}\right)}{G(r)} + \frac{\frac{d^2\phi}{d\theta^2}}{\phi(\theta)} = 0$$

$$\underbrace{\frac{r\frac{d}{dr}\left(r\frac{dG}{dr}\right)}{G(r)}}_{f^n \text{ of } r} = \underbrace{-\frac{\frac{d^2\phi}{d\theta^2}}{\phi(\theta)}}_{f^n \text{ of } \theta} = \lambda : \text{ separation constant}$$

$$\left\{\begin{array}{l} \dfrac{d^2\phi}{d\theta^2} + \lambda\phi = 0 \\ \\ \phi(-\pi) = \phi(\pi) \\ \dfrac{d\phi}{d\theta}(-\pi) = \phi(\pi) \end{array}\right.$$

: we expect oscillations in $\phi \Rightarrow \lambda > 0$

(but we also include $\lambda = 0$)

Here we can use the result for problem w/ periodic BCs and $L = \pi$, $x \to \theta$

e'values: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$, $n = 0, 1, \ldots$

e'functions: $\phi_n(\theta) = \begin{cases} \cos\frac{n\pi}{L}\theta = \cos\frac{n\pi\theta}{\theta} = \cos n\theta & n = 0, 1 \ldots \\ \sin\frac{n\pi\theta}{L} = \sin\frac{n\pi\theta}{\pi} = \sin n\theta & n = 1, 2 \ldots \end{cases}$

i.e. $\lambda_n = n^2$, $n = 0, 1, \ldots$

$$\phi_n(\theta) = \begin{cases} \cos n\theta, & n = 0, 1 \ldots \\ \sin n\theta, & n = 1, 2 \ldots \end{cases}$$

if $n \neq 0 \Rightarrow \cos n\theta, \sin n\theta$

$n = 0 \Rightarrow \lambda_0 = 0$,

$\phi_0 = 1$

G-equation

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G(r) = 0$$

$$r\left(\frac{dG}{dr} + r\frac{d^2G}{dr^2}\right) - \lambda G(r) = 0$$

$$\left\{\begin{array}{l} r^2\dfrac{d^2G}{dr^2} + r\dfrac{dG}{dr} - \lambda G(r) = 0 \\ |G(0)| < \infty \end{array}\right.$$

: equipotential or Euler eq^n (each term has the same power of r as order of derivative of G)

Assume $G(r) = r^p$.

$$\frac{dG}{dr} = p\,r^{p-1} \qquad \frac{d^2G}{dr^2} = p(p-1)\,r^{p-2}$$

$$r^2 \cdot p(p-1)\cdot r^{p-2} + r\cdot p\,r^{p-1} - \lambda r^p = 0$$

$$\underbrace{r^p}_{\neq 0}\left[\, p(p-1) + p - \lambda \,\right] = 0$$

$\therefore \quad p(p-1) + p - \lambda = 0 \quad$: characteristic eq^{n}

$$p^2 - \cancel{p} + \cancel{p} - \lambda = 0$$

$$p^2 = \lambda \qquad \Rightarrow \quad p = \pm\sqrt{\lambda}$$

$\therefore \quad r^{\sqrt{\lambda}}$ and $r^{-\sqrt{\lambda}}$ are two linearly independent solutions, provided $\lambda \neq 0$.

$\lambda_n = n^2, \quad n = 0, 1, \ldots$

$n \neq 0 \qquad \lambda_n = n^2, \quad n = 1, 2, \ldots \quad \Rightarrow \quad p = \pm n, \quad \cancel{r^n},$

Solutions are r^n and r^{-n}

$$G(r) = C_1 r^n + C_2 r^{-n}$$

$n=0 \Rightarrow \lambda_0 = 0 \Rightarrow$ only one solution w/ $p=0$, i.e. $r^0 = 1$

we need another lin. independent solution.

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \cancel{\lambda}^{\,0} G = 0$$

$$\underset{\neq 0}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \Rightarrow \frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \Rightarrow r\frac{dG}{dr} = \bar{C}_2 = \text{const}$$

$$\frac{dG}{dr} = \frac{\bar{C}_2}{r} \Rightarrow G(r) = \bar{C}_2 \underset{r>0}{\ln r} + \bar{C}_1 = \bar{C}_2 \ln r + \bar{C}_1 \cdot 1$$

$\therefore$ $\ln r$ is another lin. indep. solution for case $\lambda = 0$.

Summary for $G(r)$:

$$G(r) = \begin{cases} C_1 r^n + C_2 r^{-n}, & \lambda_n \neq 0 \quad \text{or} \quad n \neq 0 \\ \bar{C}_1 + \bar{C}_2 \ln r, & \lambda_0 = 0 \quad \text{or} \quad n = 0 \end{cases}$$

$\lambda_n = n^2$

$n = 0, 1, 2 \ldots$

BC at $r=0$: $\quad |G(0)| < \infty \quad \Rightarrow \quad c_2 = \bar{c}_2 = 0$

$$\therefore \quad G(r) = \begin{cases} c_1 r^n, & n \neq 0 \\ \bar{c}_1, & n = 0 \end{cases} \quad \equiv \quad c_1 r^n, \quad n = 0, 1, \ldots$$

Using superposition principle,

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

Note: solution is oscillatory in θ and not oscillatory in r.

BC at $r=a$: $\quad u(a,\theta) = f(\theta)$: given temperature at boundary $r=a$

$$\underset{f(\theta)}{\overset{\shortparallel}{u(a,\theta)}} = \sum_{n=0}^{\infty} \underbrace{A_n a^n} \cos n\theta + \sum_{n=1}^{\infty} \underbrace{B_n a^n} \sin n\theta$$

Fourier coefficients

we use orthogonality of $\{\cos n\theta, \sin n\theta\}$ to find $A_n a^n$, $B_n a^n$, and then solve for A_n, B_n.

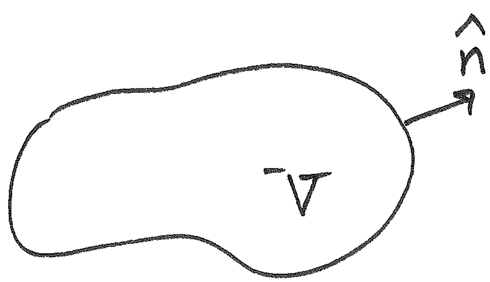
$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \qquad n=0$$

(average temperature distribution at the boundary)

$$\left.\begin{aligned} A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \end{aligned}\right\} \quad n \geq 1$$

Application: Fluid Flow past a Circular Cylinder

Before we study this problem, we will derive equation that governs conservation of mass or continuity equation.



V: arbitrary volume of fluid

$\hat{n}$: outward unit normal

$d\vec{S}$: surface element in the direction of $\hat{n}$:

$$d\vec{S} = \hat{n}\, dS$$

$\vec{u}$: fluid velocity

Normal component of $\vec{u}$ is $\vec{u} \cdot \hat{n} \parallel d\vec{S}$

Only normal component $\vec{u} \cdot \hat{n}$ of velocity transfers mass out/in volume V.

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Outward mass flux (mass flux per unit volume) through the surface element is

$$\rho \vec{u} \cdot d\vec{S} = \rho \vec{u} \cdot \hat{n} \, dS \qquad \rho: \text{density}$$

Rate of loss of mass from $\bar{V}$ is $\iint_{\partial \bar{V}} \rho \vec{u} \cdot \hat{n} \, dS$