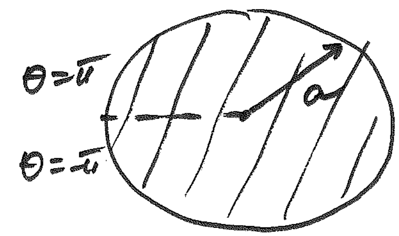
Lecture 11

Because of symmetry, we will use polar coordinates.

steady state $\rightarrow u_t = 0$



$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 \le r \le a, \quad -\pi \le \theta \le \pi$$

$u(a,\theta) = f(\theta)$: prescribed temperature

$|u(0,\theta)| < \infty$: boundedness at the origin

periodic BCs
$$\begin{cases} u(r,-\pi) = u(r,\pi) \\ u_\theta(r,-\pi) = u_\theta(r,\pi) \end{cases}$$

Separation of variables: $u(r,\theta) = \phi(\theta)\,G(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2}\cdot G(r) = 0 \quad \Big| \; \frac{1}{\frac{1}{r^2}\phi(\theta)G(r)}$$

$$\underbrace{\frac{r\frac{d}{dr}\left(r\frac{dG}{dr}\right)}{G(r)}}_{\text{function of } r} = -\underbrace{\frac{\frac{d^2\phi}{d\theta^2}}{\phi(\theta)}}_{\text{function of } \theta} = \lambda$$

$\Phi'' + \lambda\Phi = 0 \quad \Rightarrow \quad \lambda > 0$ to have an oscillatory in θ solution Φ

<u>Φ equation</u>

$$\frac{d^2\Phi}{d\theta^2} + \lambda\Phi = 0$$

$$\Phi(-\pi) = \Phi(\pi)$$

$$\frac{d\Phi}{d\theta}(-\pi) = \frac{d\Phi}{d\theta}(\pi)$$

Set $L = \pi$

e'values: $\left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$

$n = 0, 1, 2, \ldots$

e'functions: $\left[\begin{array}{l} \cos\frac{n\pi\theta}{L} = \cos\frac{n\cancel{\pi}\theta}{\cancel{\pi}} = \cos n\theta \quad n = 0, 1, 2 \ldots \\ \sin\frac{n\pi\theta}{L} = \sin\frac{n\cancel{\pi}\theta}{\cancel{\pi}} = \sin n\theta \quad n = 1, 2, \ldots \end{array}\right.$

or $\lambda_n = n^2$ $\quad n = 1, 2, \ldots \qquad \cos n\theta, \sin n\theta \to \Phi$

$n = 0 \quad \lambda = 0 \qquad 1 = \Phi$

G equation

$$r \frac{d}{dr}\left(r \frac{dG}{dr}\right) - \lambda G(r) = 0$$

$$r\left(\frac{dG}{dr} + r\frac{d^2G}{dr^2}\right) - \lambda G(r) = 0$$

$$\begin{cases} r^2 \dfrac{d^2G}{dr^2} + r\dfrac{dG}{dr} - \lambda G(r) = 0 \\ |G(0)| < \infty \end{cases}$$

: equipotential or Euler eq^{n}

Assume that $G(r) = r^p$.

$$\Rightarrow \frac{dG}{dr} = p r^{p-1} \qquad \frac{d^2G}{dr^2} = p(p-1) r^{p-2}$$

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - \lambda r^p = 0$$

$$\underbrace{r^p}_{\neq 0}\left[p(p-1) + p - \lambda\right] = 0$$

$\therefore \quad p(p-1)+p-\lambda=0$: characteristic eq^n

$p^2-\cancel{p}+\cancel{p}-\lambda=0 \Rightarrow p^2=\lambda \Rightarrow p=\pm\sqrt{\lambda}$

solutions are $r^{\sqrt{\lambda}}$ and $r^{-\sqrt{\lambda}}$

$\lambda_n = n^2, \quad n=0,1,2,\ldots$

$n \neq 0 \Rightarrow$ solutions are r^n and $r^{-n} \Rightarrow G(r)=C_1 r^n + C_2 r^{-n}$

$n=0 \Rightarrow$ we have only 1 solution: $r^0=1$

$\Rightarrow$ we need another solution

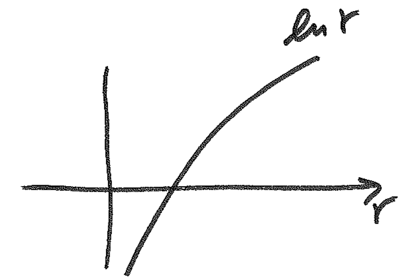
$\lambda=0$ Consider $r\frac{d}{dr}\left(r\frac{dG}{dr}\right)-\underset{=0}{\lambda} G=0 \Rightarrow r\frac{d}{dr}\left(r\frac{dG}{dr}\right)=0$

$\Rightarrow \frac{d}{dr}\left(r\frac{dG}{dr}\right)=0 \Rightarrow r\frac{dG}{dr}=\underset{\text{const}}{\bar{C}_2} \Rightarrow \frac{dG}{dr}=\frac{1}{r}\bar{C}_2 \Rightarrow G(r)=\bar{C}_2 \ln r+\underset{\text{const}}{\bar{C}_1}$

$\therefore \ln r$ is another solution in the case when $\lambda=0$.

Summary for $G(r)$:

$$G(r) = \begin{cases} C_1 r^n + C_2 r^{-n}, & n \neq 0 \\ \bar{C}_1 + \bar{C}_2 \ln r, & n = 0 \end{cases}$$



BC at $r=0$: $|G(0)| < \infty$

$$\Rightarrow C_2 = \bar{C}_2 = 0$$

Hence, $G(r) = C_1 r^n, \quad n = 0, 1, 2, \ldots$

Solution:

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

Note: solution is oscillatory in θ and it is not oscillatory in r

BC at $r=a$: $\quad u(a,\theta) = f(\theta)$: prescribed temperature at $r=a$, i.e. at the boundary of the disk

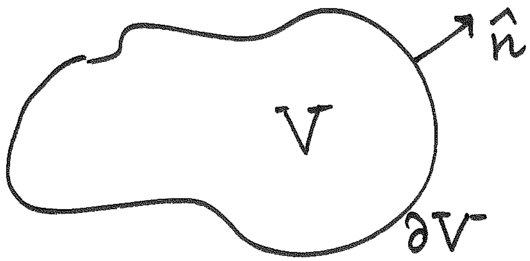
$$f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} \underbrace{A_n a^n}\cos n\theta + \sum_{n=1}^{\infty} \underbrace{B_n a^n}\sin n\theta$$

We use orthogonality conditions to find $A_n a^n$ and $B_n a^n$, and then one can solve for A_n and B_n.

$$\boxed{\begin{aligned} &A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta && n=0 \\ &\left.\begin{aligned} A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \end{aligned}\right\} && n\geq 1 \end{aligned}}$$

Application: Fluid Flow past a Circular Cylinder

Before we study this problem, let's derive equation that govern conservation of mass or continuity equation.



V: arbitrary volume of fluid (completely within fluid)

$\hat{n}$: outward unit normal

$d\vec{S}$: surface element in the direction of $\vec{n}$, i.e.

$$d\vec{S} = \vec{n}\, dS$$

$\vec{u}$: fluid velocity

Normal component of velocity is $\vec{u} \cdot \vec{n} \parallel d\vec{S}$.

Only normal component $\vec{u} \cdot \hat{n}$ of velocity $\vec{u}$ transfers mass out of volume V.

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Outward mass flux (mass flux per unit time) through surface element is

$$\rho\vec{u}\cdot d\vec{S} = \rho\vec{u}\cdot\hat{n}\,dS$$

Here, ρ is fluid density.

Rate of loss of mass from V is $\iint_{\partial V} \rho\vec{u}\cdot\hat{n}\,dS$

<u>Note</u>: flux $> 0 \Rightarrow$ mass $\searrow$

flux $< 0 \Rightarrow$ mass $\nearrow$

Total mass inside V is $\iiint_V \rho\, dV$

Law of mass conservation:

$$\frac{d}{dt}\iiint_V \rho\, dV = -\iint_{\partial V} \rho\vec{u}\cdot\hat{n}\,dS$$