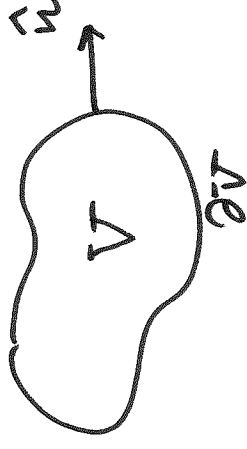


Recall divergence (Gauss) theorem:

$$\iiint_V \underbrace{\nabla \cdot \vec{A}}_{\text{div } \vec{A}} dV = \iint_{\partial V} \vec{A} \cdot \hat{n} dS$$



$$\iiint_{\partial V} \underbrace{\rho \vec{u} \cdot \hat{n}}_{\text{div.}} dS = \frac{d}{dt} \iiint_V \nabla \cdot (\rho \vec{u}) dV$$

Hence, mass conservation law becomes

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right] dV = 0$$

Since \$V\$ is arbitrary

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$

mass conservation law in
a differential form

$$\vec{A} = \langle A_1, A_2, A_3 \rangle$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\frac{d}{dt} \iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Fluid: gas or liquid

$\rho = \rho(x, y, t)$: fluid density

$\vec{u} = \vec{u}(x, y, t)$: velocity

Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$

$$\nabla \cdot (\rho \vec{u}) = \nabla \rho \cdot \vec{u} + \rho \nabla \cdot \vec{u}$$

$$\therefore \underbrace{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u}}_{\text{mass that moves with fluid}} = 0 \quad \left\{ \begin{array}{l} \text{density change} \\ \text{due to compression} \end{array} \right.$$

Water: incompressible

Air: compressible

Ideal Fluid Assumptions

1. Incompressibility: $\nabla \cdot \vec{u} = 0$

$$\vec{u} = (u_1, u_2) \Rightarrow$$

$$\boxed{\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0}$$

continuity equation

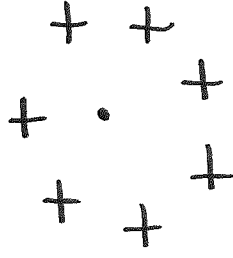
cannot change fluid density

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

2. Fluid is irrotational: $\nabla \times \vec{u} = 0$

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{k} & \hat{j} & \hat{i} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & 0 \end{vmatrix} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \hat{k}$$

$$\Rightarrow \boxed{\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0}$$

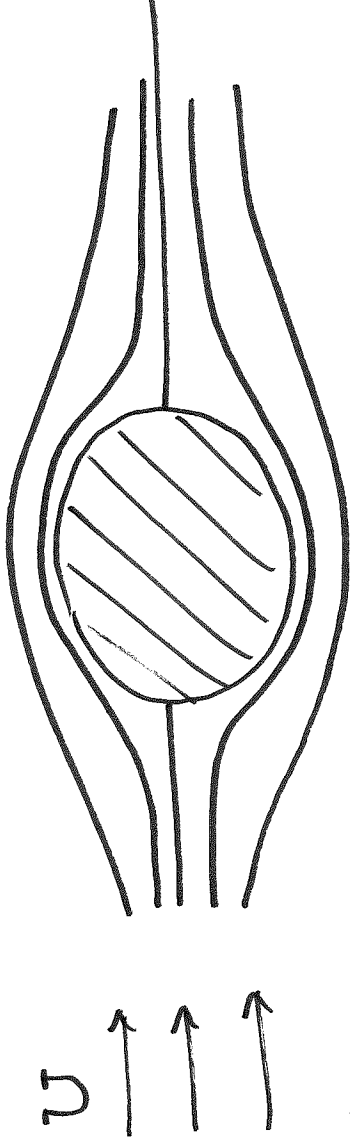


Introduce stream function

$$\psi = \psi(x, y)$$

Streamlines are curves
where $\psi = \text{const}$

Streamlines are parallel
to the fluid flow:
at any pt (x, y) ,
tangent vector of a
streamline is parallel



to the fluid velocity at (x, y) .

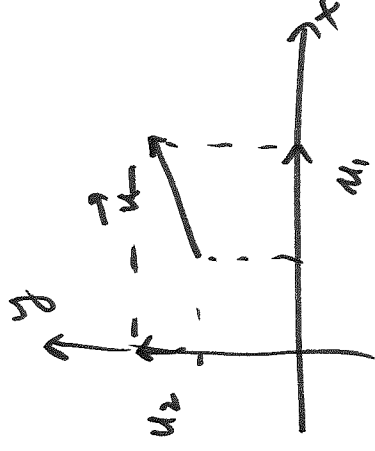
We define stream function ψ :

$$u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = -\frac{\partial \psi}{\partial x}$$

Aside: Fluid potential $\phi = \phi(x, y)$
is defined as a function:

$$u_1 = \frac{\partial \phi}{\partial x}, \quad u_2 = \frac{\partial \phi}{\partial y}$$

$$\vec{u} = \nabla \phi$$



Fluid is irrotational \Rightarrow

$$\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0$$

$$u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\therefore \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \right] \quad \text{or} \quad \boxed{\nabla^2 \psi = 0}$$

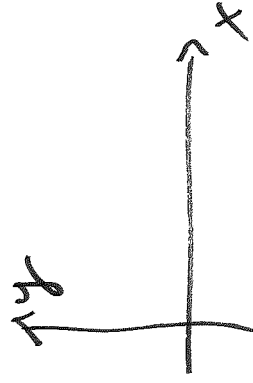
Laplace's eqⁿ

\therefore Stream function ψ satisfies Laplace's equation, i.e. $\psi(x, y)$ is harmonic function.

Flow past the circular cylinder



We have a uniform flow very far from the disk in the horizontal direction.



$(u_1, u_2) = (U, 0)$: velocity far away from disk

$\nabla^2 \psi = 0$ outside a circular domain

Because of circular geometry, it is convenient to use polar coordinates. Laplacian in polar coordinates has the form

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

We will use

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad \psi = \psi(r, \theta)$$

For $r \gg 1$ (very far from cylinder) we have

$$u_1 = U, \quad u_2 = 0$$

$$u_1 = \frac{\partial \psi}{\partial y} = U$$

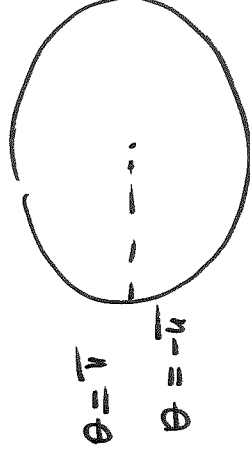
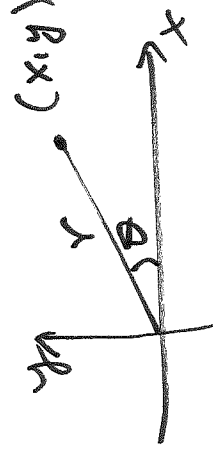
$$\Rightarrow \psi = yU$$

$$u_2 = -\frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \psi(y) \text{ only}$$

Polar coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$\therefore \psi \approx yU = r \sin \theta \cdot U \text{ for } r \gg 1$$

Problem can now be written as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

$$\psi(a, \theta) = 0$$

$$\psi \approx r \sin \theta \cdot U \text{ for } r \gg 1$$

Periodic

$$\psi(r, -\pi) = \psi(r, \pi)$$

BCs in θ :

$$\psi_{\theta}(r, -\pi) = \psi_{\theta}(r, \pi)$$

Separation of variables: $\psi(r, \theta) = G(r) \phi(\theta)$

Solution (see the problem about steady-state flow inside circular disk - see previous lecture 11 2/6/2017):

$$\psi(r, \theta) = \bar{C}_2 + \bar{C}_1 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \sin n\theta + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \cos n\theta \quad (1)$$

BCs

$$1. \psi(a, \theta) = 0$$

$$\bar{C}_2 + \bar{C}_1 \ln a + \sum_{n=1}^{\infty} (A_n a^n + B_n a^{-n}) \sin n\theta + \sum_{n=1}^{\infty} (C_n a^n + D_n a^{-n}) \cos n\theta = 0$$

From orthogonality of cosines and sines or the fact that $\{\cos n\theta, \sin n\theta\}$ is a linearly independent set, we get

$$\bar{C}_2 + \bar{C}_1 \ln a = 0 \Rightarrow \bar{C}_2 = -\bar{C}_1 \ln a$$

$$A_n a^n + B_n a^{-n} = 0 \Rightarrow B_n = -A_n a^{2n}$$

$$C_n a^n + D_n a^{-n} = 0 \Rightarrow D_n = -C_n a^{2n}$$

$$\bar{C}_2 + \bar{C}_1 \ln r = -\bar{C}_1 \ln a + \bar{C}_1 \ln r = \bar{C}_1 \ln \frac{r}{a}$$

$$A_n r^n + B_n r^{-n} = A_n r^n - A_n a^{2n-n} r^{-n} = A_n \left(r^n - \frac{a^{2n}}{r^n} \right)$$

similarly for $C_n a^n + D_n r^{-n} = C_n \left(r^n + \frac{a^{2n}}{r^n} \right)$

$$\therefore \psi(r, \theta) = \bar{C} \ln \frac{r}{a} + \sum_{n=1}^{\infty} A_n \left(r^n - \frac{a^{2n}}{r^n} \right) \sin n\theta + \sum_{n=1}^{\infty} C_n \left(r^n + \frac{a^{2n}}{r^n} \right) \cos n\theta$$