For $r \gg 1$, $\psi(r,\theta) \approx \underbrace{r\sin\theta}_{y} \cdot U$

$r \gg 1 \Rightarrow r^n \gg \ln r, \quad n \geq 1$

$r^n \gg \frac{1}{r^n}$

$$\Rightarrow \psi \approx \sum_{n=1}^{\infty} A_n r^n \sin n\theta + \sum_{n=1}^{\infty} C_n r^n \cos n\theta = r\sin\theta \cdot U$$

$\therefore \quad A_1 = \bar{U}, \quad A_n = 0, \; n \geq 2$

$C_n = 0, \quad n \geq 1$

Hence,

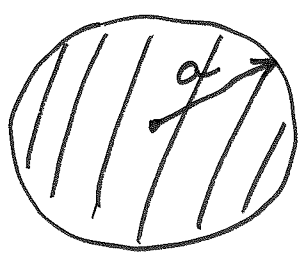
$$\boxed{\psi(r,\theta) = \bar{C}_1 \ln\frac{r}{a} + U\left(r - \frac{a^2}{r}\right)\sin\theta} \qquad r \geq a$$

Note Solution is not unique since it depends on an arbitrary constant $\bar{C}_1$ (we will discuss this later).

Fluid pressure p exerts force that acts in the direction opposite to outward normal $\hat{n}$ to the cylinder:

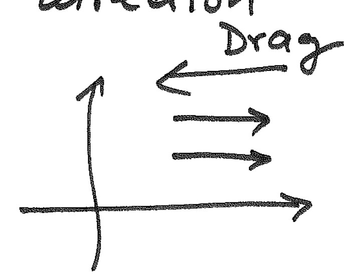
$$\vec{F} = -\int_C p(a,\theta)\,\hat{n}\,dl = -\int_0^{2\pi} p(a,\theta)\,\hat{n}\cdot a\,d\theta$$

$$dl = r\,d\theta$$

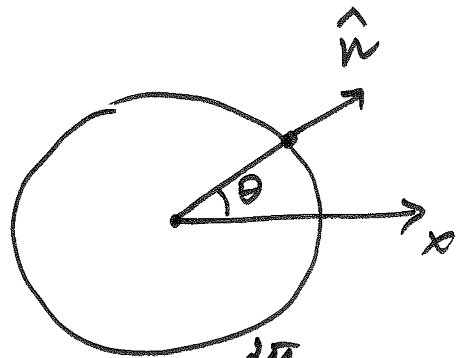


$$p = p(r,\theta)$$

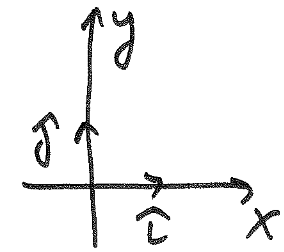
Drag is a horizontal component of $\vec{F}$ in the direction opposite to fluid motion.



Lift is a vertical component of $\vec{F}$ (in positive y direction).



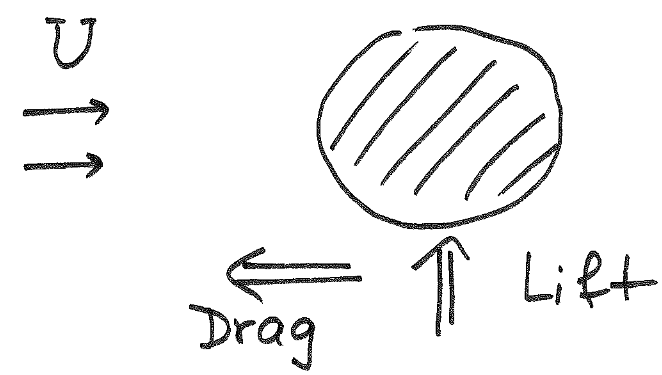
$$\hat{n} = \cos\theta \cdot \hat{\imath} + \sin\theta \cdot \hat{\jmath} = \langle \cos\theta, \sin\theta \rangle$$



$$\text{Lift} = -\int_0^{2\pi} \left(p(a,\theta) \cdot \hat{n} \right) \cdot \hat{\jmath} \; a \, d\theta \; =$$

$$\hat{n} \cdot \hat{\jmath} = \left(\cos\theta \cdot \hat{\imath} + \sin\theta \cdot \hat{\jmath} \right) \cdot \hat{\jmath} = \sin\theta$$

$$= -\int_0^{2\pi} p(a,\theta) \sin\theta \cdot a \, d\theta \quad \boxed{=}$$

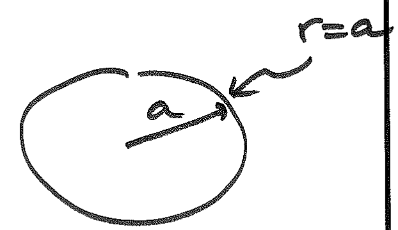


Bernoulli's condition:

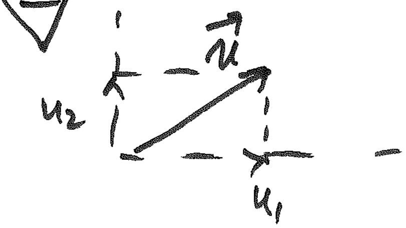
$$p + \frac{1}{2} \rho |\vec{u}|^2 = \text{const} = p_0 \quad \text{(for steady flow)}$$

$$\Rightarrow \; p = p_0 - \frac{1}{2} \rho |\vec{u}|^2$$

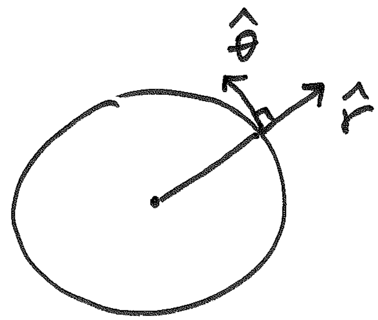
$$\boxed{\equiv} - \int_0^{2\pi} \left(p_0 - \frac{1}{2}\rho |\vec{u}|^2 \right)\Bigg|_{r=a} \sin\theta \cdot a\, d\theta =$$



$$= -\int_0^{2\pi} p_0 a \cdot \cancel{\sin\theta\, d\theta}^{\,0} + \frac{1}{2}\int_0^{2\pi} \rho |\vec{u}|^2 \Big|_{r=a} \cdot \sin\theta \cdot a\, d\theta \;\boxed{=}$$



assume $\rho = \text{const.}$



$\vec{u} = u_1 \cdot \hat{i} + u_2 \cdot \hat{j}$

$\vec{u} = u_r \cdot \hat{r} + u_\theta \cdot \hat{\theta}$

not partial derivatives, this is just notation

$\vec{u} = u_r \cdot \hat{r} + u_\theta \cdot \hat{\theta}$

$|\vec{u}|^2 = u_r^2 + u_\theta^2$

$u_\theta = -\dfrac{\partial \psi}{\partial r} \qquad u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \qquad \psi$: streamfunction

We obtained earlier today solution

$$\psi(r,\theta) = \bar{C}_1 \ln\frac{r}{a} + U\left(r - \frac{a^2}{r}\right)\sin\theta, \quad r \geq a$$

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$$u_\theta = -\frac{\partial\psi}{\partial r} = -\bar{C}_1 \frac{1}{r} - \bar{U}\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\bar{U}}{r}\left(r - \frac{a^2}{r}\right)\cos\theta = \bar{U}\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$|\vec{u}|^2\Big|_{r=a} = \cancelto{0}{u_r^2\Big|_{r=a}} + u_\theta^2\Big|_{r=a} = \left(\frac{\bar{C}_1}{a} + 2\bar{U}\sin\theta\right)^2$$

$$\boxed{=} \; \frac{\rho a}{2}\int_0^{2\pi} \sin\theta\left(\left(\frac{\bar{C}_1}{a}\right)^2 + 4\frac{\bar{C}_1}{a}\bar{U}\sin\theta + 4\bar{U}^2\sin^2\theta\right)d\theta =$$

$$= \frac{\rho a}{2}\left[\int_0^{2\pi}\left(\frac{\bar{C}_1}{a}\right)^2 \cancelto{0}{\sin\theta\, d\theta} + 4\frac{\bar{C}_1}{a}U\int_0^{2\pi}\sin^2\theta\, d\theta + 4\bar{U}^2\int_0^{2\pi}\cancelto{0}{\underbrace{\sin^3\theta}_{odd}}\, d\theta\right] =$$

$$= \frac{\rho a}{2}\cdot 4\frac{\bar{C}_1}{a}U\underbrace{\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\, d\theta}_{\frac{1}{2}\cdot 2\pi} = 2\pi\rho\bar{C}_1 U = \underline{Lift}$$

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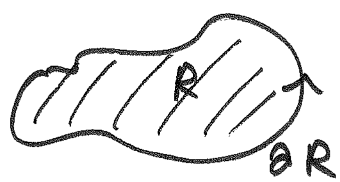
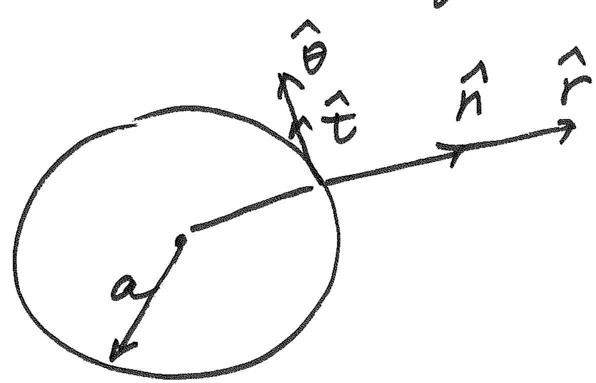
$\therefore \quad Lift = 2\pi \rho \bar{c}_1 U$

$Lift > 0 \quad \text{if} \quad \bar{c}_1 > 0.$

unit tangent vector

<u>Circulation</u> $\quad \Gamma = \oint_{\partial R} \vec{u} \cdot d\vec{l} = \oint_{\partial R} \underbrace{\vec{u} \cdot \hat{t}}_{u_\theta} \underbrace{dl}_{r\,d\theta} \; =$





On the circle of radius a

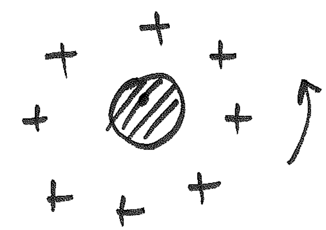
$r\,d\theta = a\,d\theta$

$$= \int_0^{2\pi} u_\theta \cdot a\,d\theta \quad \boxed{=}$$

$$\vec{u} = u_r \cdot \hat{r} + u_\theta \cdot \hat{\theta}$$

$$u_\theta = -\frac{\bar{c}_1}{r} - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

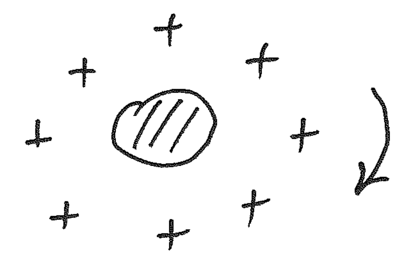
$$\equiv \int_0^{2\pi} a\left(-\frac{\bar{C}_1}{a} - U\underbrace{\left(1+\frac{a^2}{a^2}\right)}_{=2}\cancel{\sin\theta}^{\,0}\right) d\theta = -\bar{C}_1 \cdot 2\pi$$



$\Gamma > 0$

$$\Rightarrow \boxed{\Gamma = -2\pi\bar{C}_1}$$

Circulation is positive if motion is in counterclockwise direction. Since to have lift we need $\bar{C}_1 > 0$, circulation $\Gamma < 0$, hence, motion is in clockwise direction. One can show that in this case, velocity on the top of cylinder is higher than velocity on the bottom and as a result, the pressure is lower on top than on the bottom, which creates lift.



$\Gamma < 0$