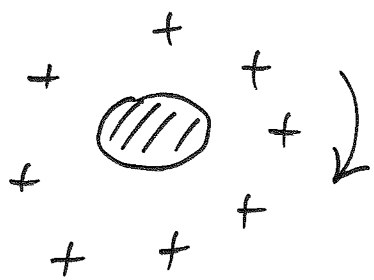
<u>Note</u> When circulation is taken over infinitesimal loop, it gives <u>vorticity</u> per unit area.



Orientation of the trace + will not change since flow is irrotational.

The character of streamlines depends on the value of circulation Γ.

We found

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

where $\Gamma = -2\pi C_1$: circulation around the cylinder, $\Gamma < 0$.

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<u>Def</u> <u>Stagnation points</u> are points where velocity is zero.

$$\vec{u} = u_r \cdot \hat{r} + u_\theta \cdot \hat{\theta}$$

$$\vec{u} = 0 \Rightarrow u_r = 0, \quad u_\theta = 0$$

$$u_r = 0 \Rightarrow \underset{\neq 0}{U}\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0 \Rightarrow r = a \quad \text{or} \quad \cos\theta = 0$$

$$0 \le \theta < 2\pi \Rightarrow \cos\theta = 0 \quad \text{when} \quad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2}$$

$$u_\theta = 0 \Rightarrow -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} = 0$$

Let $\underset{>0}{B} = -\frac{\Gamma}{2\pi U a} = \frac{\Gamma}{2\pi}\left(-\frac{1}{Ua}\right) \Rightarrow \frac{\Gamma}{2\pi r} = -\frac{BUa}{r}$

$$\Rightarrow u_\theta = 0 \Rightarrow -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{BUa}{r} = 0$$

$$-\underset{\neq 0}{U}\left[\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{Ba}{r}\right] = 0$$

$$\Rightarrow \underbrace{\left(1 + \frac{a^2}{r^2}\right)}_{>0}\sin\theta + \overbrace{\frac{Ba}{r}}^{>0} = 0 \Rightarrow$$ $\sin\theta < 0$, i.e. θ is in the lower half plane

Conditions for stagnation points are:

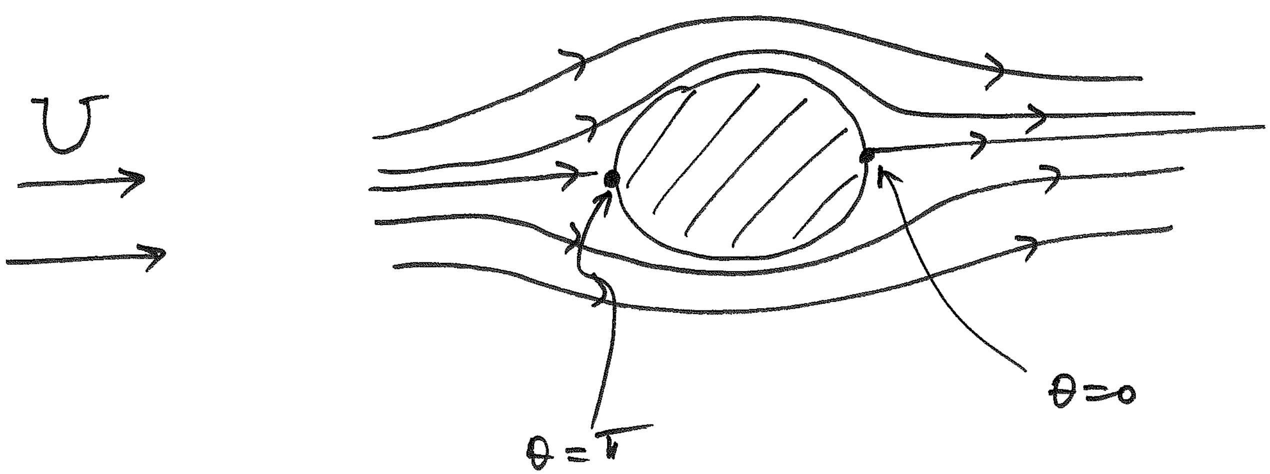
$$r = a \quad \text{or} \quad \theta = \frac{3\pi}{2}$$

<u>Case 1</u>: $r = a \Rightarrow$ stagnation points are on the cylinder

$$u_r\Big|_{r=a} = 0 \qquad u_\theta\Big|_{r=a} = -\underset{\neq 0}{U}\left[2\sin\theta + B\right] = 0$$

$$\Rightarrow 2\sin\theta + B = 0 \quad \text{or} \quad \sin\theta = -\frac{B}{2}$$

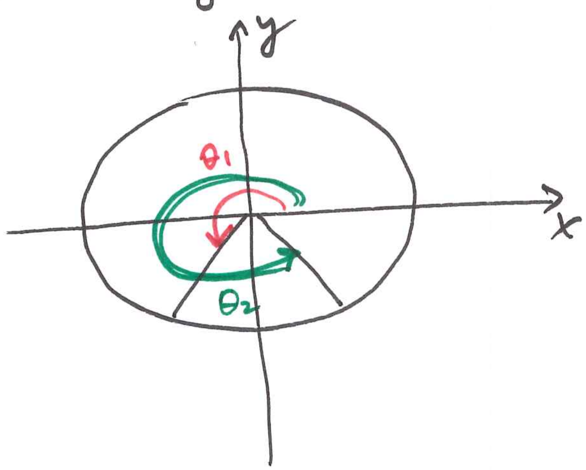
<u>Case (a)</u>: $\boxed{B = 0} \Rightarrow \Gamma = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or $\theta = \pi$

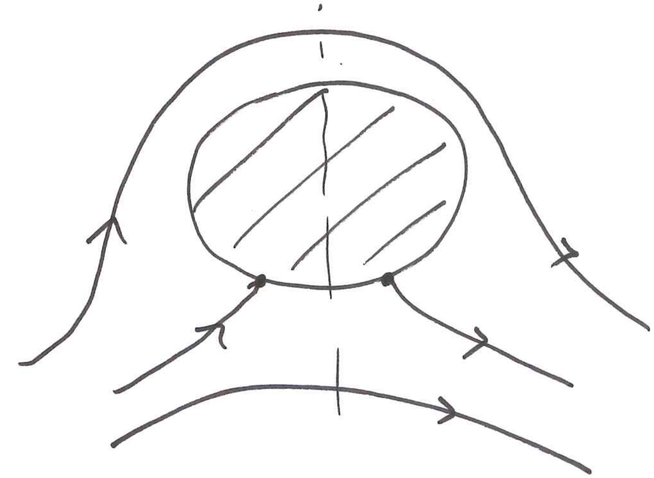


Case (b): $0 < B < 2$

$B = -2\sin\theta \Rightarrow 0 < -2\sin\theta < 2$ or $-1 < \sin\theta < 0$

two stagnation points at $r = a$ and $\theta = \theta_1$, $\theta = \theta_2$

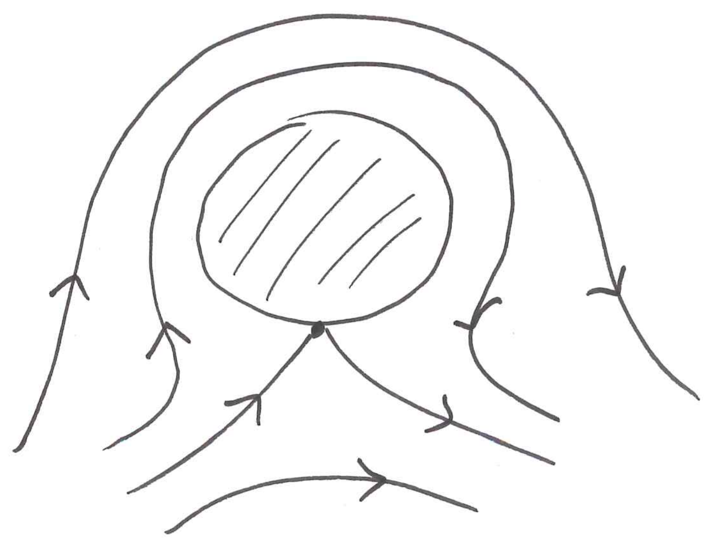




Case (c): $B = 2 \Rightarrow \sin\theta = -1$

$\Rightarrow \theta = \frac{3\pi}{2}$

One stagnation pt at $r = a$, $\theta = \frac{3\pi}{2}$



Case (d): $B > 2$: no real solutions

Case 2: $\theta = \frac{3\pi}{2} \Rightarrow \cos\theta = 0 \Rightarrow u_r = 0$

$$u_\theta = -U_0 \left[-\left(1 + \frac{a^2}{r^2}\right) + \frac{Ba}{r} \right] = 0 \rightarrow [\ldots] = 0$$

$$\Rightarrow r^2 - Bar + a^2 = 0$$

Two roots:

$$r = \frac{Ba \pm \sqrt{(Ba)^2 - 4a^2}}{2}$$

or

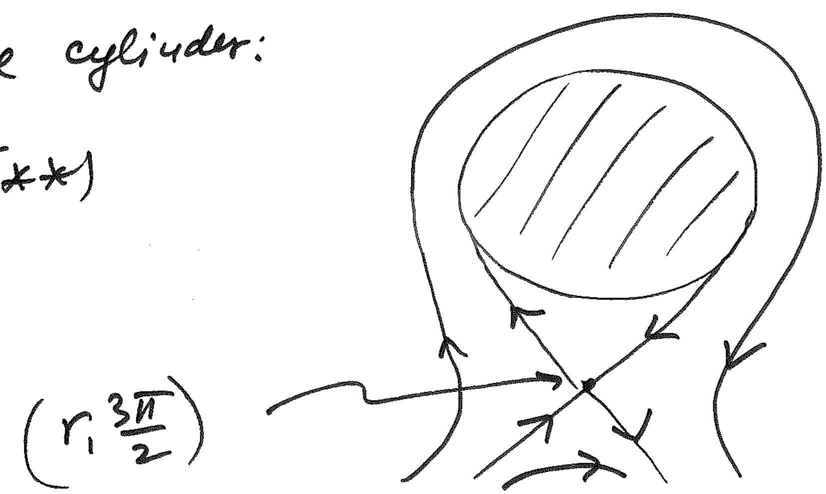
$$\frac{r}{a} = \frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 - 1}$$

$B > 2 \Rightarrow \frac{r}{a} = \frac{B}{2} - \sqrt{\left(\frac{B}{2}\right)^2 - 1} < 1 \Rightarrow r < a \Rightarrow$ point is inside cylinder $\lightning$

But the second root is outside the cylinder:

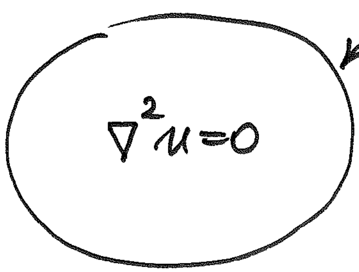
$$\frac{r}{a} = \frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 - 1} \qquad (**)$$

We have one stagnation pt with $\theta = \frac{3\pi}{2}$ and r given by $(**)$



Properties of Laplace's equation

Recall the problem about steady state heat flow inside the circular domain.



$u(a,\theta) = f(\theta)$: prescribed temperature

We obtained solution valid for $r \le a$:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n r^n \cos n\theta + B_n r^n \sin n\theta \right]$$

$$u(0,\theta) = A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$\therefore$ Temperature at $r=0$ equals the average value of temperature at the edge of the cirle.

<u>Def</u> Functions that satisfy Laplace's equation are called <u>harmonic functions</u>.

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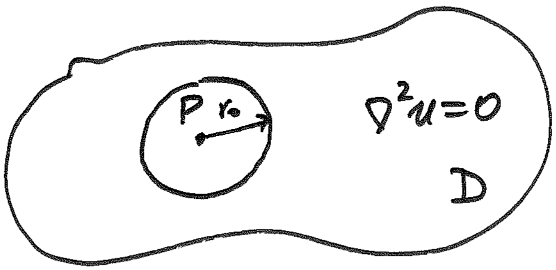
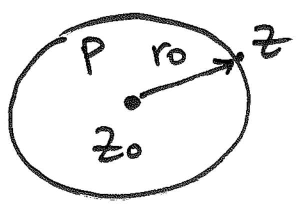
More generally, consider a domain D and assume that $\nabla^2 u = 0$ in D.



Take an arbitrary pt $P \in D$ and a circle w/ center at P and radius r_0 such that this circle is within domain D. Then

$$u(P) = \text{average value of } u(r_0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_0,\theta)\,d\theta$$



$$z = z_0 + r_0 e^{i\theta}$$

$$u(z) = u(r_0,\theta)$$

$-\pi \le \theta < \pi$

points on the circle centered at P and radius r_0

This is a characteristic property of harmonic functions.

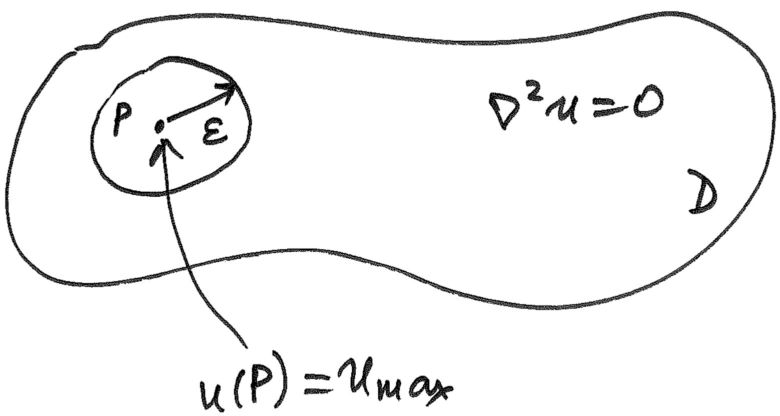
Maximum principle

The solution of $\nabla^2 u = 0$ in domain D attains its maximum on the boundary of D (unless $u \equiv const$).

Proof

Ignore the case $u \equiv const$.

Assume that u achieves its max value at some pt P inside D. Enclose this point with a circle of radius ε so that circle is inside D.



By characteristic property of harmonic functions

$$u(P) = u_{max} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\varepsilon, \theta)\, d\theta$$

average value of u on the circle centered at P w/ rad $= \varepsilon$

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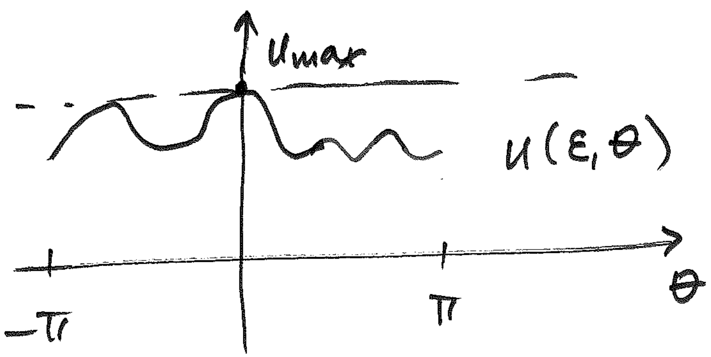
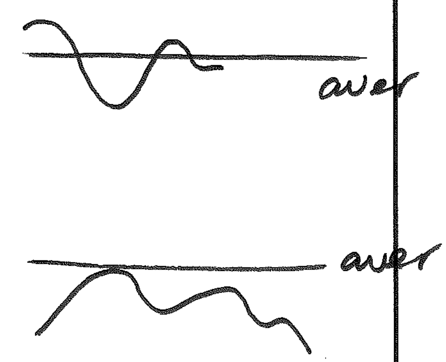
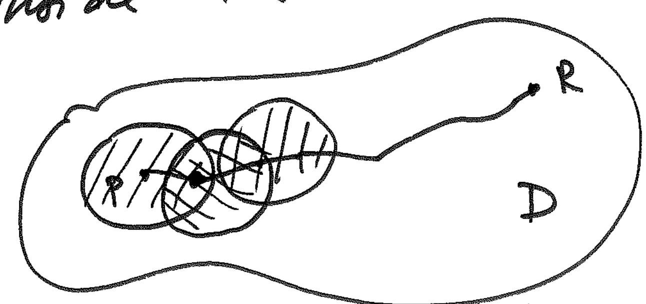


all values on the circle
$u(\varepsilon, \theta) \le u_{max}$

The strict equality is achieved when
$u(\varepsilon, \theta) = u_{max}$ for all θ



This is true for any $\varepsilon \Rightarrow u = u_{max} = const$
inside the disk centered at pt P.



Take any other point that is inside the circle with center at P. The value at that other point is also u_{max}, and we can repeat the previous reasonings. In this way, we connect any point R in D with pt P by a broken line. The value inside of these disks is u_{max} and $u(R) = u_{max}$. Since one can do this for any point inside D, this shows that $u = u_{max} = const$ everywhere in D. ↯

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<u>Note</u> To prove that a harmonic function achieves its min on the boundary, use $-u(r,\theta)$ instead.

<u>Well-posedness</u>

<u>Def</u> We say that a problem is <u>well-posed</u> if there exists a unique solution that continuously depends on the nonhomogeneous data [small changes in data produce small changes in a solution].

Problem is <u>well-posed</u> if

1) solution exists
2) solution is unique
3) solution continuously depends on data

a problem that is not well-posed is called <u>ill-posed</u>.

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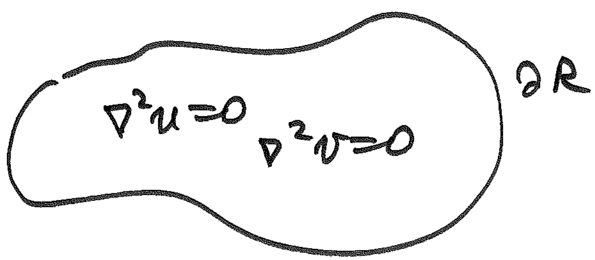
BVP with Dirichlet BC

Consider a region R



$\nabla^2 u = 0 \quad \text{in } R \qquad u\big|_{\partial R} = f$

$\nabla^2 v = 0 \quad \text{in } R \qquad v\big|_{\partial R} = g$

Let $f \approx g$.