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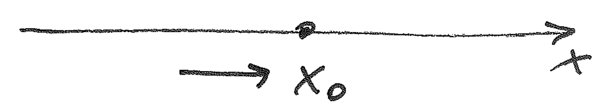
Lecture 15

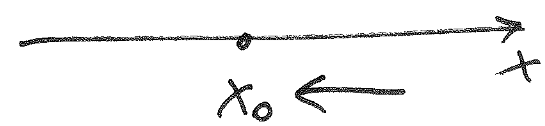
**Def** a <u>jump discontinuity</u> occurs at pt $x = x_0$ if the limit from the left

$$f(x_0^-) = \lim_{x \to x_0^-} f(x)$$




exists (i.e. finite) and the limit from the right.

$$f(x_0^+) = \lim_{x \to x_0^+} f(x)$$



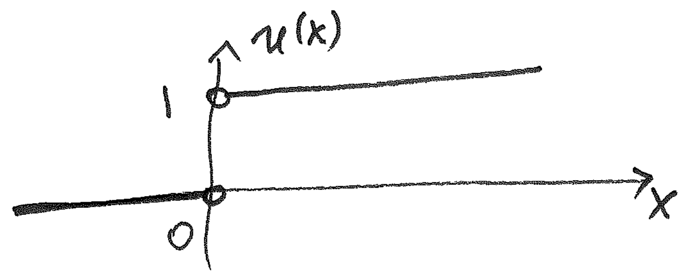

also exists BUT

$$f(x_0^-) \neq f(x_0^+)$$

We define <u>jump</u> at $x_0$:

$$[f(x_0)] = f(x_0^+) - f(x_0^-)$$

Ex

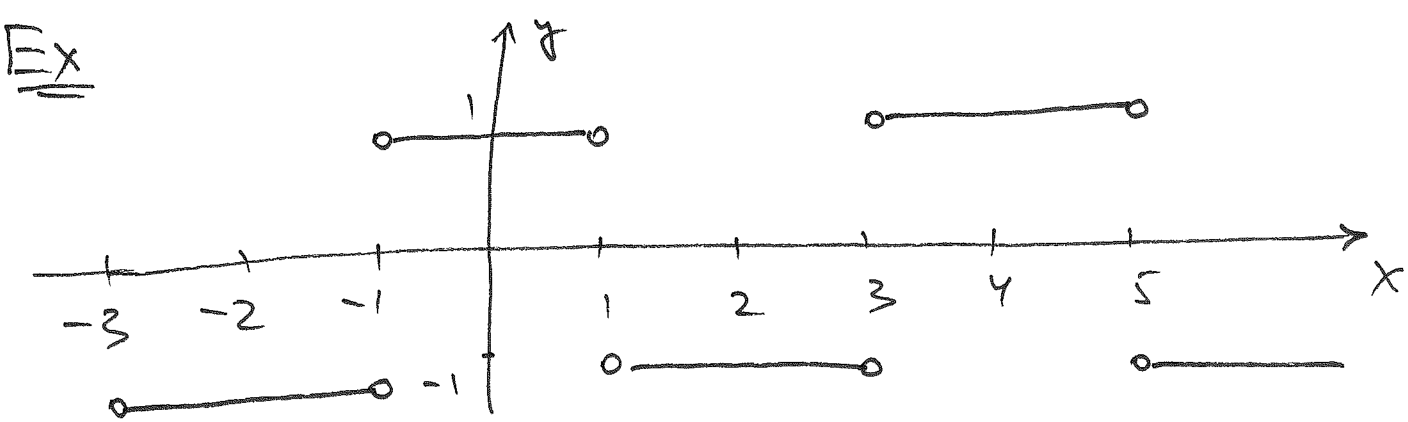



unit step function

$$u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$[f(0)] = f(0^+) - f(0^-) = 1 - 0 = 1$$

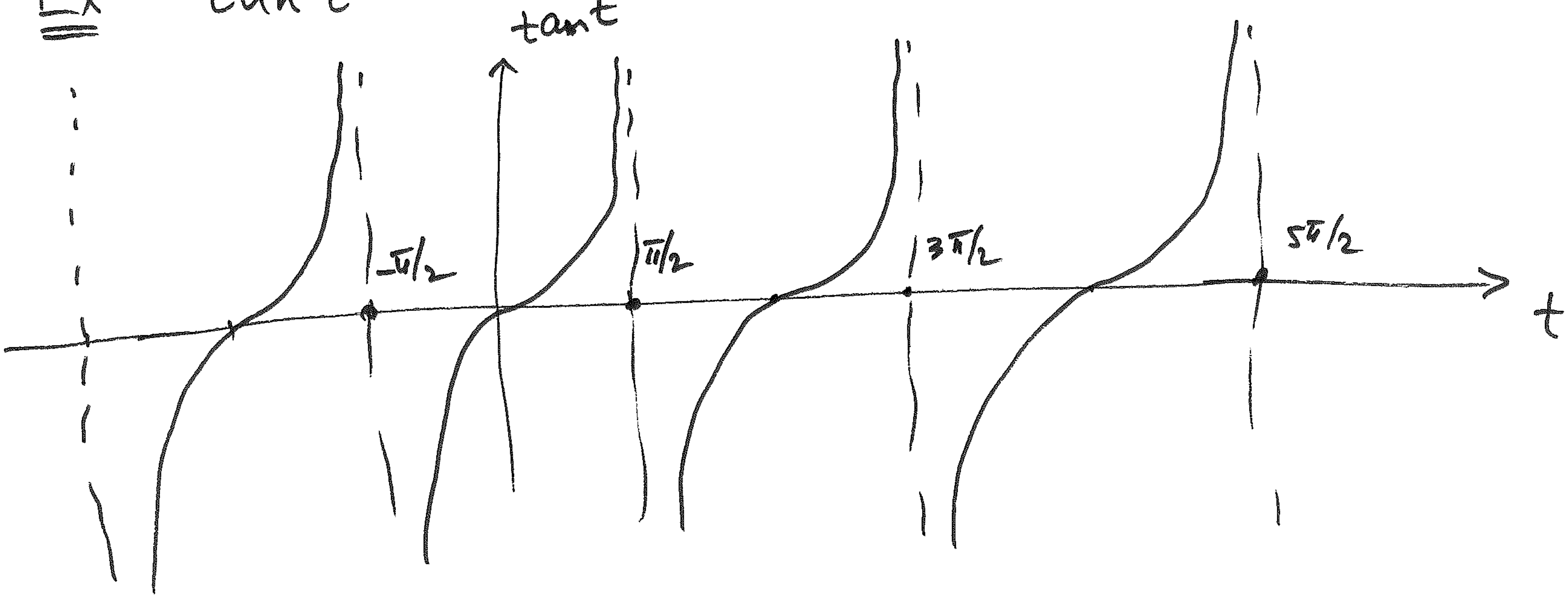
Ex



jump discontinuities are at $x_n = 2n+1$, $n \in \mathbb{Z}$

$$[f(x_n)] \doteq \pm 2$$

$\underline{\underline{\text{Ex}}}$ $\tan t$




$\tan t$ has <u>infinite</u> jump discontinuities at $\frac{\pi}{2} + \pi n$

$\therefore$ $\tan t$ is <u>NOT</u> piecewise cts. or piecewise smooth

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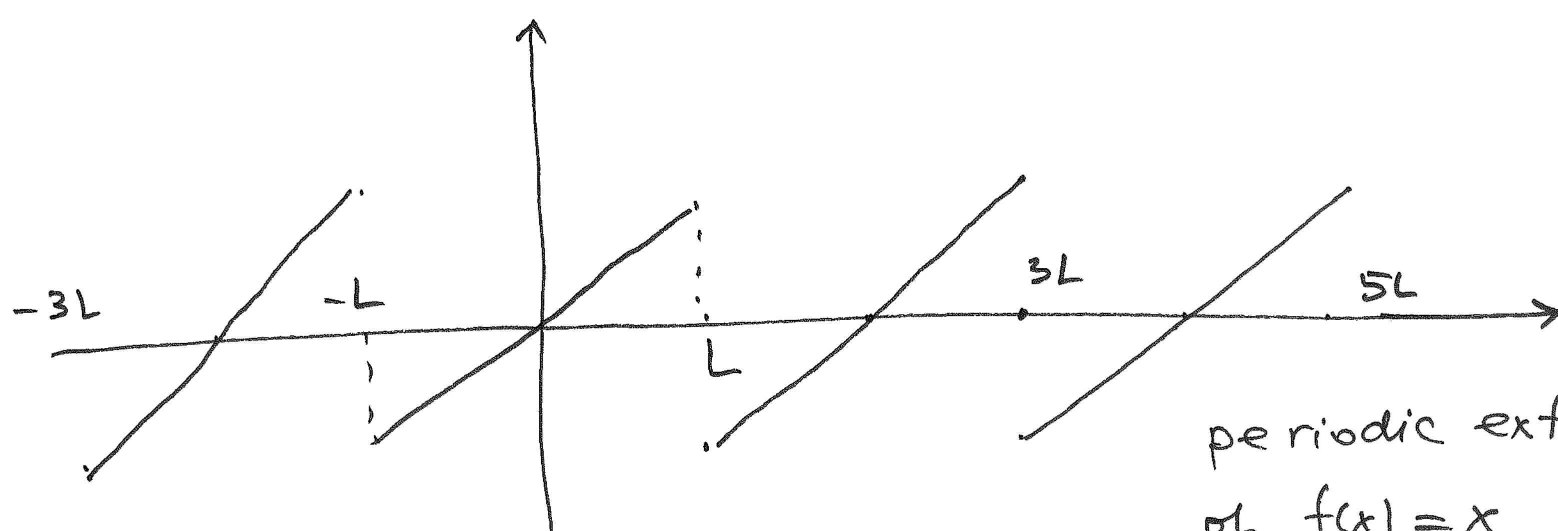
**Def** Let $f(x)$ be defined on $-L \le x \le L$. The *periodic extension* of $f(x)$ is obtained by "copying" and "pasting" $f(x)$ into the intervals

$-L + 2nL \le x \le L + 2nL$, where $n = \pm 1, \pm 2, \pm 3, \ldots$

**Ex** $f(x) = x$, $\quad -L \le x \le L$




periodic extension of $f(x) = x$

It is $2L$-periodic

<u>Note</u> if $f(x)$ is $2L$-periodic, then its periodic extension is the function itself.

<u>Def</u> The <u>Fourier series</u> of a function $f(x)$ over an interval $-L \le x \le L$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

and the <u>Fourier coefficients</u> are (using orthogonality of sines and cosines):

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$n \ge 1$

**Note** We use symbol "$\sim$" to say that $f(x)$ <u>has</u> a Fourier series, but this Fourier series may not converge or if it converges, it may not converge to $f(x)$ (it might converge to a different function).

## Fourier Thm

If $f(x)$ is piecewise smooth on the interval $-L \le x \le L$, then the Fourier series converges to:

1) periodic extension of $f(x)$, where periodic extension is continuous;

2) average of the two limits:
$$\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$$

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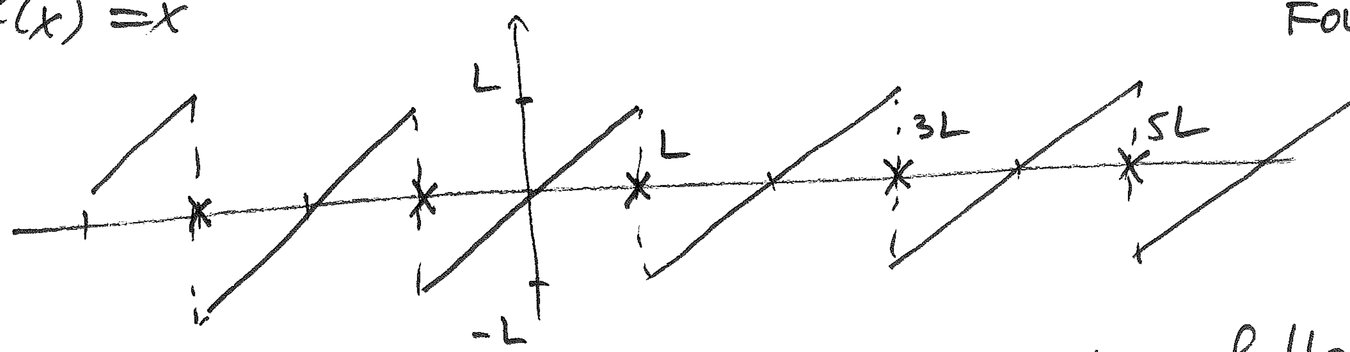
where periodic extension has a finite jump discontinuity.

**Ex** $f(x) = x$




sketch of Fourier series of $f(x)=x$

We can use Fourier Thm to write the following.

Let $f(x)$ be piecewise smooth on $-L \leq x \leq L$. Then

(a) if $f(x)$ is continuous at $x = x_0 \in (-L, L)$, then

$$f(x_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x_0}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x_0}{L}$$

(b) if $f(x)$ has a finite jump discontinuity at $x = x_0$, then

$$\frac{1}{2}\left[ f(x_0^-) + f(x_0^+) \right] = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x_0}{L}$$

(c) if $x_0 = L$ or $x_0 = -L$

$$x_0 = L \Rightarrow \quad \cos\frac{n\pi x_0}{L} = \cos\frac{n\pi L}{L} = \cos n\pi = (-1)^n$$

$$\sin\frac{n\pi x_0}{L} = \sin\frac{n\pi L}{L} = \sin n\pi = 0$$

$$x_0 = -L \Rightarrow \quad \cos\frac{n\pi x_0}{L} = \cos\frac{n\pi(-L)}{L} \underset{\text{even } f^n}{\overset{\cos \text{ is}}{=}} \cos\frac{n\pi L}{L} = (-1)^n$$

$$\sin\frac{n\pi x_0}{L} = \sin\frac{n\pi(-L)}{L} = 0$$

Then

$$\frac{1}{2}\left[ f(L) + f(-L) \right] = a_0 + \sum_{n=1}^{\infty} a_n (-1)^n$$

If $x_0$ is outside $[-L, L]$, then replace $f(x)$ with its periodic extension.

## Sketching Fourier Series

1. Sketch $f(x)$ on $-L < x < L$
2. Sketch periodic extension of $f(x)$.
3. Mark an "x" at the average of two one-sided limits of $f(x)$ where $f(x)$ has a finite jump discontinuity.

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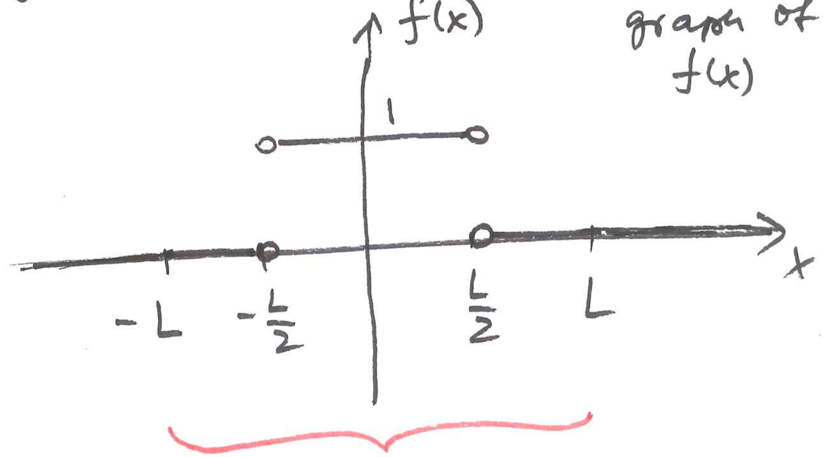
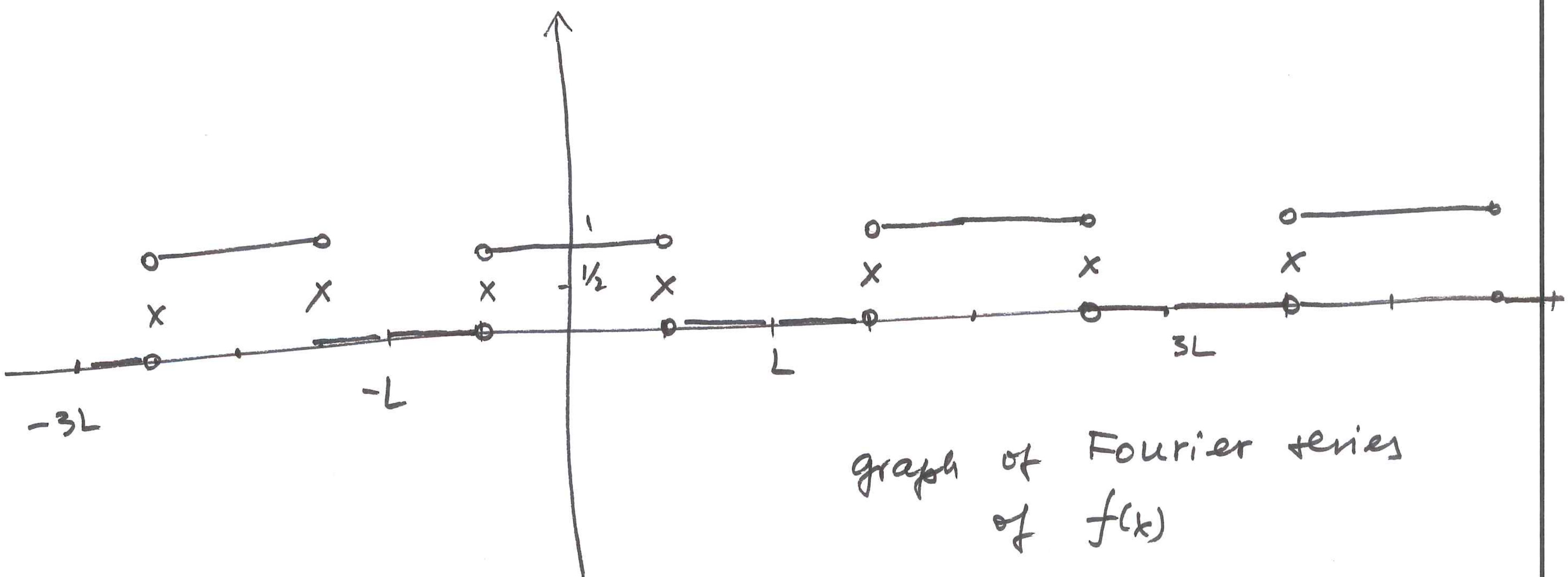
Ex Sketch Fourier series for

$$f(x) = \begin{cases} 1 & |x| < \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$




graph of f(x)




graph of Fourier series of f(x)

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$\therefore$

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} 1, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & -L \le x < -\frac{L}{2} \\ 0, & \frac{L}{2} < x \le L \\ \frac{1}{2}, & x = \pm\frac{L}{2} \end{cases}$$

We will compute Fourier coefficients next time.