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Lecture 15

$\nabla^2 u = 0$ in R $\quad u\big|_{\partial R} = f$

$\nabla^2 v = 0$ in R $\quad v\big|_{\partial R} = g$

Let $f \approx g$. Introduce $w = u - v$. By linearity

$\nabla^2 w = 0$ in R and $w\big|_{\partial R} = f - g$

Using the maximum/minimum principle

$$\min\{f-g\} = w_{min} \le w \le w_{max} = \max\{f-g\}$$

Since $f \approx g \Rightarrow f - g \approx 0 \Rightarrow w \approx 0 \Rightarrow u - v \approx 0$ or $u \approx v$

i.e. small change in data produces a small change in the solution.

Show <u>uniqueness</u> of the solution. Proof by contradiction

$\nearrow$ Assume that there are two solutions u and v that satisfy the same BVP:

$$\nabla^2 u = 0 \quad \text{in } R \quad \text{and} \quad u\big|_{\partial R} = f$$

$$\nabla^2 v = 0 \quad \text{in } R \quad \text{and} \quad v\big|_{\partial R} = f$$

Introduce $w = u - v$. Then

$$\nabla^2 w = 0 \quad \text{in } R \quad \text{and} \quad w\big|_{\partial R} = f - f = 0$$

By Maximum principle,

$$0 = w_{min} \leq w \leq w_{max} = 0 \quad \Rightarrow \quad w = 0$$

$\therefore\ u - v = 0 \Rightarrow u = v$ ↯

Therefore, BVP with Dirichlet BCs is well-posed.

Now, let's consider BVP with Neumann BCs.
We prescribe the heat flux on the boundary:

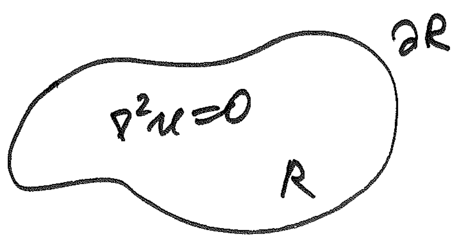
$$-K_0 \nabla u \cdot \hat{n} = f(x), \qquad x \in \partial R$$

Consider BVP:

$$\nabla^2 u = 0 \qquad \text{in } R$$

$$-K_0 \nabla u \cdot \hat{n} = f(x) \quad \text{on} \quad x \in \partial R$$

Q When will this problem have a solution?



$$0 = \iint_R \underbrace{\nabla^2 u}_{=0} \, dx\,dy = \iint_R \nabla \cdot (\nabla u)\, dx\,dy \underset{\text{Thm}}{\overset{\text{divergence}}{=}}$$

$$= \int_{\partial R} \nabla u \cdot \hat{n} \, dl \implies$$

$$\Rightarrow \boxed{\oint_{\partial R} \nabla u \cdot \vec{n}\, dl = 0}$$

total heat flux / net flux through boundary has to be $= 0$

Recall,

$$-K_0 \nabla u \cdot \vec{n} = f(x) \quad \text{on } \partial R$$

Let $K_0 = \text{const}$

$$\therefore \oint_{\partial R} \nabla u \cdot \vec{n}\, dl \Rightarrow \boxed{\oint_{\partial R} f(x)\, dl = 0}$$

solvability condition when $K_0 = \text{const}$

Therefore, for a steady-state solution of the heat problem w/ Neumann BC to exist, the total heat flux through the boundary has to be zero. This is called SOLVABILITY CONDITION.

Fourier Series (Chapter 3)

Recall: solution to the heat equation

$u_t = k u_{xx}$

w/ periodic BCs on $-L < x < L$ is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

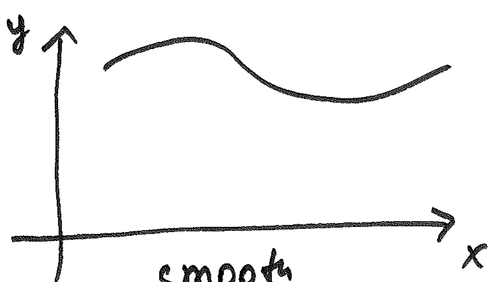
IC: $$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

This is a Fourier series.

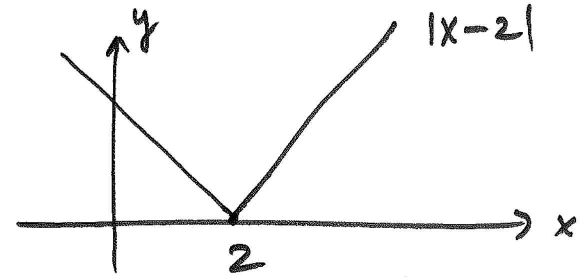
Does this infinite series converge? If it does converge, which function it converges to?

Def a function $f(x)$ is called <u>smooth</u> in $a < x < b$ if $f(x)$ and $f'(x)$ are continuous.

Note $f(x)$ is differentiable, i.e. $f(x)$ has a derivative at x $\Rightarrow$ $f(x)$ is continuous at x.



smooth



not smooth
but continuous

Def Function $f(x)$ is <u>piecewise smooth</u> in $a < x < b$ if there is a finite number of points where $f(x)$ or/and $f'(x)$ are not continuous

OR

$f(x)$ is <u>piecewise smooth</u> in $a < x < b$ if it is smooth on a finite number of subintervals of (a, b) except at a finite number of points at which f or f' have

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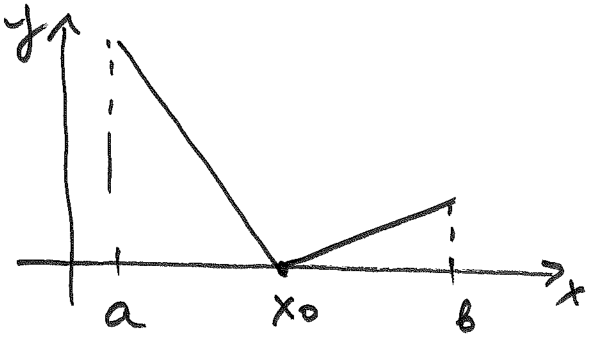
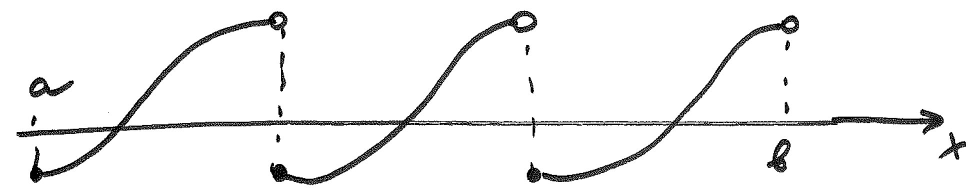
finite jumps.

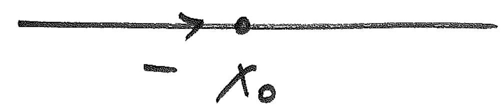


piecewise smooth



piecewise smooth

Def A jump discontinuity occurs at pt $x = x_0$ if the limit from the left



$$f(x_0^-) = \lim_{x \to x_0^-} f(x)$$

exists (i.e. finite) and the limit from the right also exists

$$f(x_0^+) = \lim_{x \to x_0^+} f(x)$$

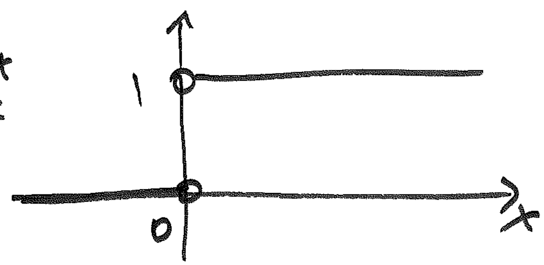
but

$f(x_0^-) \neq f(x_0^+)$

We define jump at x_0:

$$[f(x_0)] = f(x_0^+) - f(x_0^-)$$

Ex



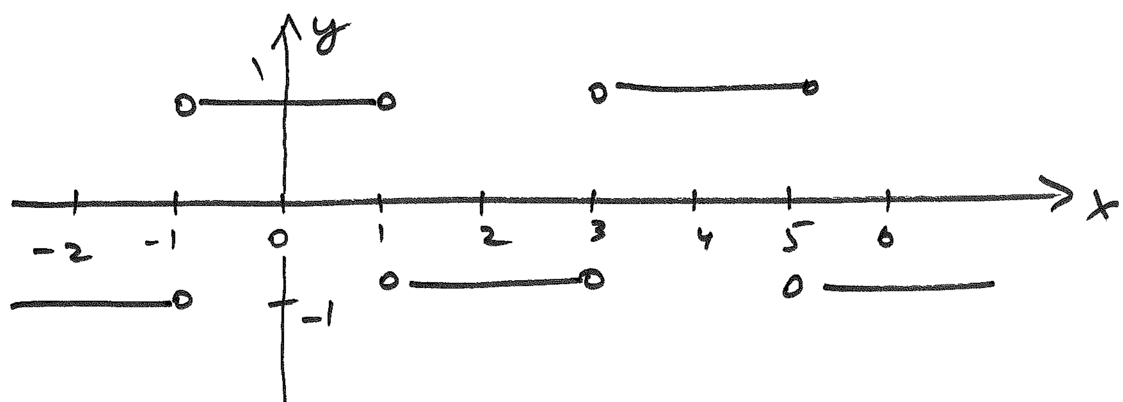
$u(x)$

unit step function

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$[f(0)] = 1 - 0 = 1$$

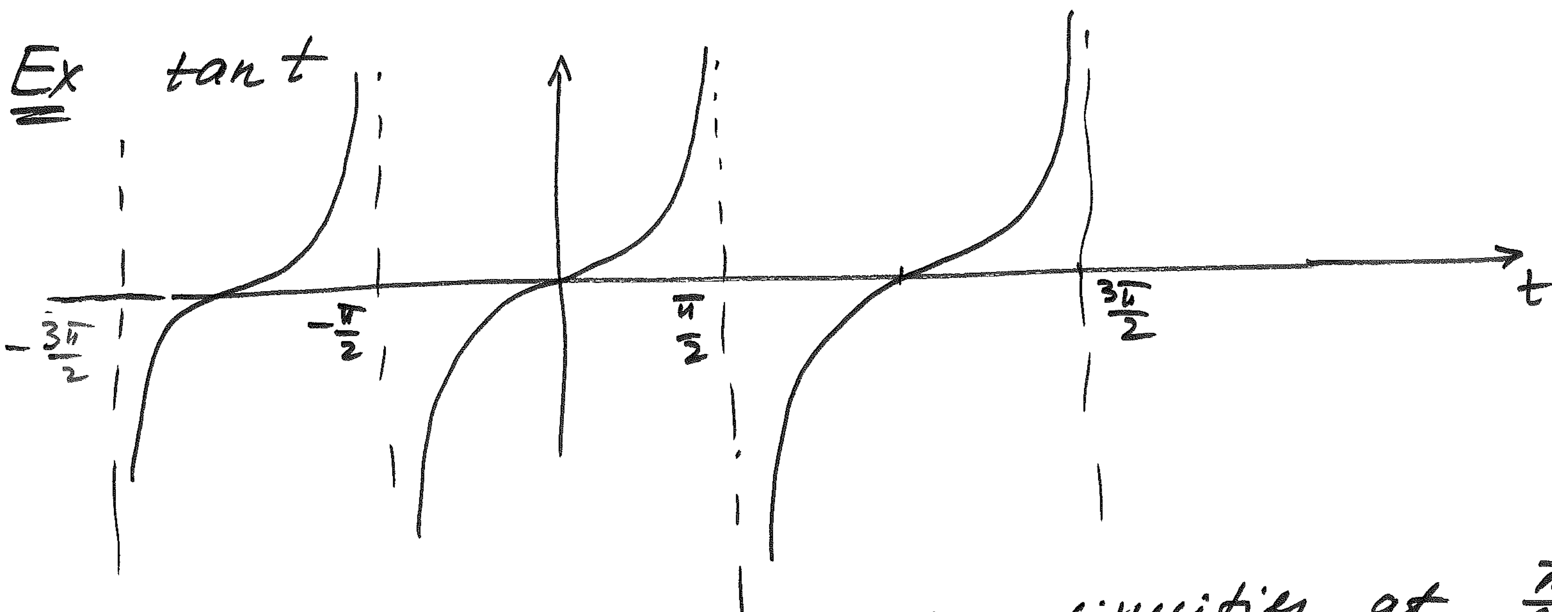
Ex



jump discontinuities are at $x = 2n+1$

$[f(\cdot)] = \pm 2$

Ex $\tan t$



$\tan t$ has infinite jump discontinuities at $\frac{\pi}{2} + n\pi$.

$\therefore$ $\tan t$ is not piecewise smooth

Def Let $f(x)$ be defined on $-L \le x \le L$. The "periodic extension" of $f(x)$ is obtained by "copying" and "pasting" $f(x)$ into the intervals $-L + 2nL \le x \le L + 2nL$, where $n = \pm 1, \pm 2, \dots$

Ex $f(x) = x, \quad -L \le x \le L$