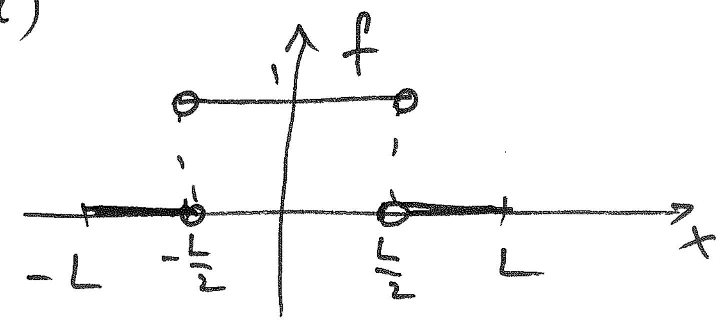
Lecture 16

(Cont'd)

Ex Fourier series for

$$f(x) = \begin{cases} 1, & |x| < \frac{L}{2} \\ 0, & |x| > \frac{L}{2} \end{cases}$$

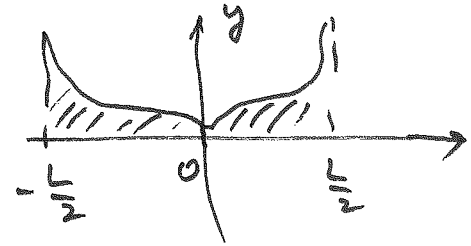


is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} 1, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & -L \leq x < -\frac{L}{2} \\ 0, & \frac{L}{2} < x < L \\ \frac{1}{2}, & x = \pm\frac{L}{2} \end{cases}$$

Now, let's compute Fourier coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left\{ \cancel{\int_{-L}^{-L/2} \underbrace{f(x)}_{=0}\,dx} + \int_{-L/2}^{L/2} \underbrace{f(x)}_{=1}\,dx + \cancel{\int_{L/2}^{L} \underbrace{f(x)}_{=0}\,dx} \right\} =$$



$$= \frac{1}{2L} \int_{-L/2}^{L/2} 1 \, dx = \frac{1}{2L} \cdot L = \boxed{\frac{1}{2}}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{-L/2}^{L/2} 1 \cdot \cos \frac{n\pi x}{L} \, dx =$$

even f^n over symmetric interval

$$= \frac{1}{L} \cdot 2 \int_0^{L/2} \cos \frac{n\pi x}{L} \, dx = \frac{2}{\cancel{L}} \cdot \frac{\cancel{L}}{n\pi} \sin \frac{n\pi x}{L} \Bigg|_{x=0}^{x=\frac{L}{2}} = \frac{2}{n\pi} \sin\left(\frac{n\pi}{L} \cdot \frac{L}{2}\right) =$$

$$= \boxed{\frac{2}{n\pi} \sin \frac{n\pi}{2}}$$

n odd $\Rightarrow$ $n = 2k+1$ $\Rightarrow$ $\sin \frac{n\pi}{2} = (-1)^k$

$n = 1, 2, \ldots$ $\quad$ $k = 0, 1 \ldots$

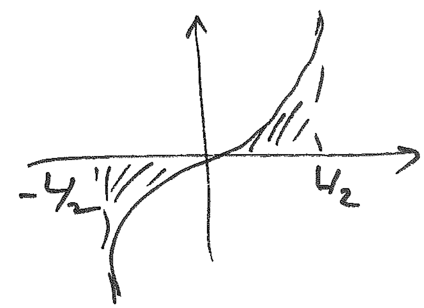
n even $\Rightarrow$ $n = 2k$ $\Rightarrow$ $\sin\frac{n\pi}{2} = \sin\frac{2k\pi}{2} = \sin k\pi = 0$

$k = 1, 2 \ldots$

$$\therefore \boxed{a_n = \begin{cases} 0, & n \text{ is even} \\ (-1)^k \cdot \dfrac{2}{n\pi}, & n \text{ is odd, ie. } n = 2k+1 \end{cases}}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L/2}^{L/2} 1\cdot\underbrace{\sin\frac{n\pi x}{L}}_{\text{odd } f^{\underline{n}}}\,dx = \boxed{0 = b_n}$$

over symmetric interval



Hence,

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}(-1)^k \cos\frac{(2k+1)\pi x}{L}$$

$n = 2k+1$

or

$$f(x) \sim \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{L}$$

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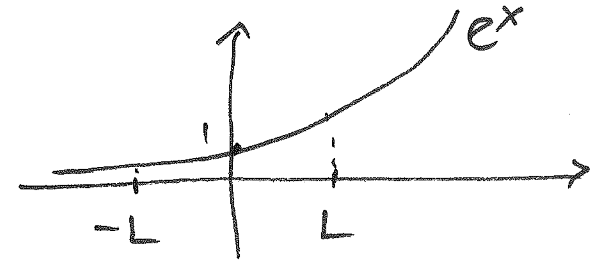
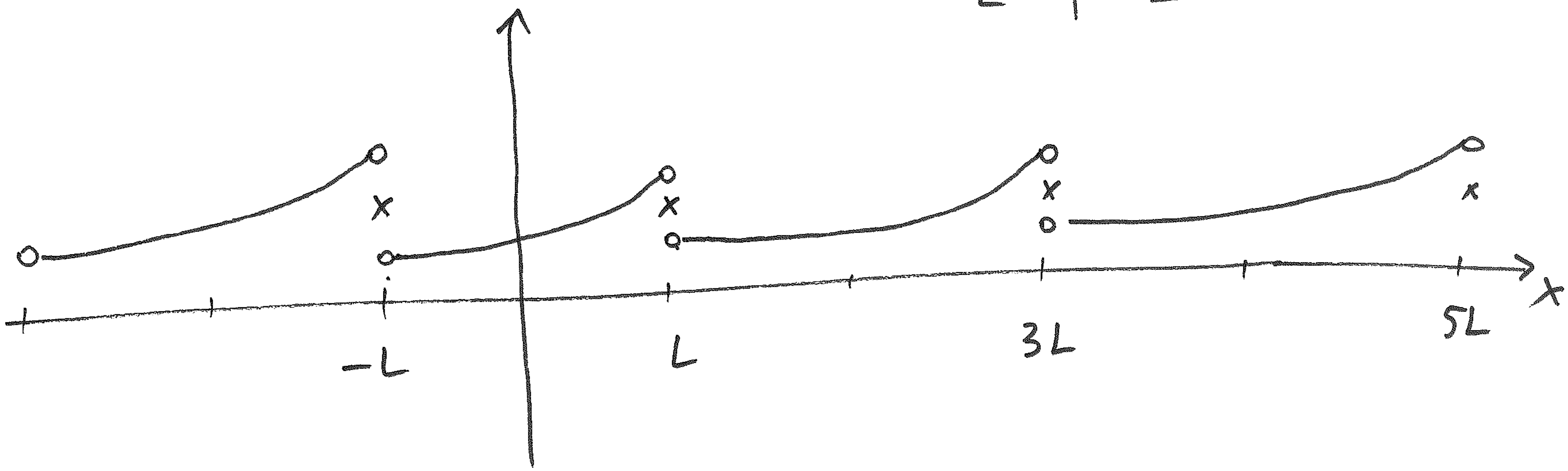
Ex $f(x) = e^x$. Sketch Fourier series and compute Fourier coefficients.





Fourier series is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} e^x, & -L < x < L \\ \frac{1}{2}\left(e^L + e^{-L}\right) & \text{at } x = \pm L \end{cases}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} e^x\,dx = \frac{1}{2L} e^x \Big|_{x=-L}^{L} = \boxed{\frac{1}{2L}\left(e^L - e^{-L}\right)}$$

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\underbrace{\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\,dx}_{=A} \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$= \left|\begin{array}{ll} u=e^x & dv=\cos\frac{n\pi x}{L}\,dx \\ du=e^x\,dx & v=\frac{L}{n\pi}\sin\frac{n\pi x}{L}\end{array}\right| = \frac{1}{L}\left\{ e^x\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} e^x\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= \left|\begin{array}{ll} u=e^x & dv=\sin\frac{n\pi x}{L}\,dx \\ du=e^x\,dx & v=-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\end{array}\right| = \frac{1}{L}\left\{\frac{L}{n\pi}e^L\underbrace{\cancel{\sin\frac{n\pi L}{L}}}_{0} - \frac{L}{n\pi}e^{-L}\underbrace{\cancel{\sin\frac{n\pi(-L)}{L}}}_{0}\right.$$

$$\left. - \frac{L}{n\pi}\left[-\frac{L}{n\pi}e^x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\,dx\right]\right\} =$$

$$= \frac{1}{L}\left\{0 - 0 - \frac{L}{n\pi}\left[-\frac{L}{n\pi}e^L\underbrace{\cos(n\pi)}_{(-1)^n} + \frac{L}{n\pi}e^{-L}\underbrace{\cos(-n\pi)}_{(-1)^n} + \frac{L}{n\pi}A\right]\right\}$$

$$\therefore \quad \frac{1}{L} A = -\frac{1}{n\pi}\left[-\frac{L}{n\pi} e^{L}(-1)^n + \frac{L}{n\pi} e^{-L}(-1)^n + \frac{L}{n\pi} A\right]$$

$$\frac{1}{L} A = -\frac{1}{n\pi} \cdot \frac{L}{n\pi}\left[(-1)^n\left(e^{-L} - e^{L}\right) + A\right]$$

$$\frac{1}{L} A = -\frac{L}{(n\pi)^2}\left[(-1)^n\left(e^{-L} - e^{L}\right) + A\right]$$

Solving for A, we get

$$\boxed{A = \frac{L^2}{L^2 + (n\pi)^2} (-1)^n \left(e^{L} - e^{-L}\right)}$$

Then

$$\boxed{a_n = \frac{1}{L} A = \frac{L}{L^2 + (n\pi)^2} (-1)^n \left(e^{L} - e^{-L}\right), \quad n \geq 1} \qquad (1)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{-L}^{L} e^{x} \sin\frac{n\pi x}{L}\, dx$$

(do integration by parts once)

One can show that

$$\boxed{b_n = -\frac{n\pi}{L} a_n}$$

$\therefore$ Fourier series is

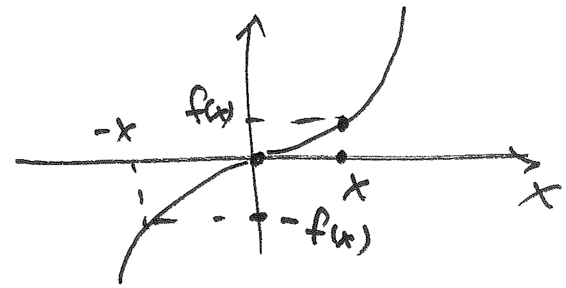
$$\underbrace{\frac{1}{2L}\left(e^{L} - e^{-L}\right)}_{a_0} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n \sin\frac{n\pi x}{L} \sim f(x)$$

where a_n is defined in (1).

3.3 Fourier Sine and Fourier Cosine Series

Def An odd function has a property

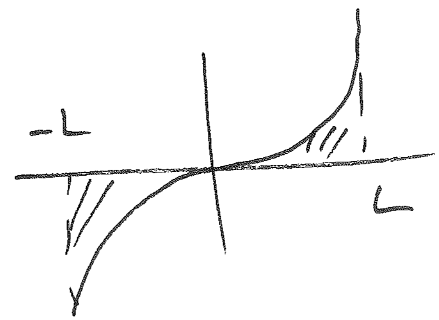
$$f(-x) = -f(x)$$



Fourier coefficients:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \underset{\text{odd}}{f(x)}\,dx = 0 \quad \text{by symmetry}$$

symmetric interval



OR

$$a_0 = \frac{1}{2L} \left\{ \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \right\} =$$

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$$\int_{-L}^{0} f(x)\,dx = \left| \begin{array}{l} u = -x \\ du = -dx \\ x=0 \Rightarrow u=0 \\ x=-L \Rightarrow u=L \end{array} \right| = \int_{L}^{0} f(-u)\,(-du) = \int_{0}^{L} \underbrace{f(-u)}_{\overset{\text{odd}}{=}\,-f(u)}\,du$$

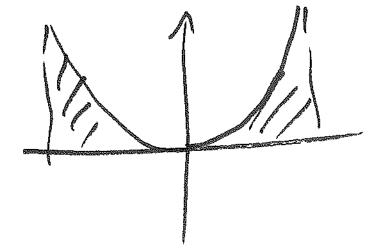
$$= -\int_{0}^{L} f(u)\,du$$

$$= \frac{1}{2L}\left\{ -\int_{0}^{L} f(u)\,du + \int_{0}^{L} f(x)\,dx \right\} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{odd}}{f(x)} \cos \underset{\text{even}}{\frac{n\pi x}{L}}}_{\substack{\text{odd} \\ \text{over symmetric interval}}}\,dx = 0, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{odd}}{f(x)}\ \underset{\text{odd}}{\sin\frac{n\pi x}{L}}}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

over symmetric interval



Hence,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} :$$

Fourier sine series

Here $f(x)$ is odd f^n

Recall: Solution to $u_t = k u_{xx}$ w/ Dirichlet BCs on $0 < x < L$ is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

IC: $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$: Fourier sine series

but $f(x)$ is ANY Function

Note: the expansion looks the same as Fourier series for an odd function but in the case of the heat equation, the initial temperature $f(x)$ was ANY FUNCTION.

Def ~~Let~~ Let $f(x)$ be a function defined on $0 \le x \le L$. The "odd extension" of $f(x)$ is obtained by "copying", "negating", "flipping" and "pasting" the result into intervals $nL \le x \le nL + L$, $n = \pm 1, \pm 2, \dots$

Ex $f(x) = x$