Lecture 16

<u>Note</u> If $f(x)$ is $2L$-periodic, then its periodic extension is the function itself.

<u>Def</u> The <u>Fourier series</u> of a function $f(x)$ over an interval $-L \le x \le L$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

and the <u>Fourier coefficients</u> are (using orthogonality of cosines and sines)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$$

$n \geq 1$

Note We use symbol "$\sim$" to say that $f(x)$ has a Fourier series, but this series may not converge or if it converges, it may not converge to $f(x)$ (it might converge to a different function).

Fourier Thm

If $f(x)$ is piecewise smooth on the interval $-L \le x \le L$, then the Fourier series converges to:

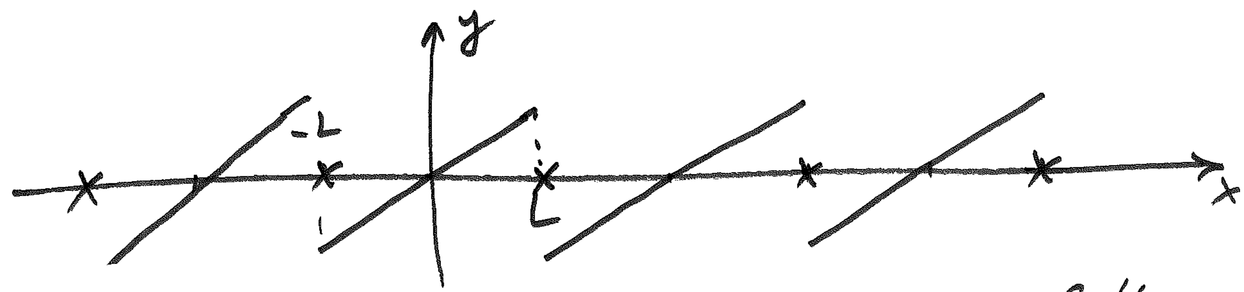
1) periodic extension of $f(x)$, where periodic extension is continuous;

2) average of the two limits:

$$\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$$

where periodic extension has a finite jump discontinuity.

Ex $f(x) = x$



We can use Fourier thm to write the following. Then

Let $f(x)$ be piecewise smooth on $-L \le x \le L$. Then

a) if $f(x)$ is continuous at $x = x_0 \in (-L, L)$, then

$$f(x_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x_0}{L}$$

(b) if $f(x)$ has a finite jump discontinuity at $x = x_0$, then

$$\frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right] = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x_0}{L}$$

University of Idaho

(c) if $x_0 = L$ or $x_0 = -L$

$$x_0 = L \Rightarrow \cos\frac{n\pi x_0}{L} = \cos\frac{n\pi L}{L} = \cos n\pi = (-1)^n$$

$$\sin\frac{n\pi x_0}{L} = \sin\frac{n\pi L}{L} = \sin n\pi = 0$$

$$x_0 = -L \Rightarrow \cos\frac{n\pi x_0}{L} = \cos\frac{n\pi(-L)}{L} = \cos(-n\pi) \overset{\text{cos is}}{\underset{\text{even } f^n}{=}} \cos(n\pi) = (-1)^n$$

$$\sin\frac{n\pi x_0}{L} = \sin\frac{n\pi(-L)}{L} = 0$$

Then

$$\frac{1}{2}[f(L) + f(-L)] = a_0 + \sum_{n=1}^{\infty} a_n (-1)^n$$

If x_0 is outside $[-L, L]$, then replace $f(x)$ with its periodic extension.

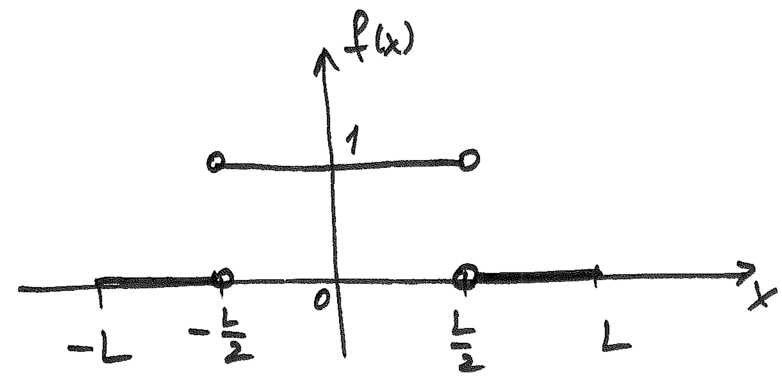
University of Idaho

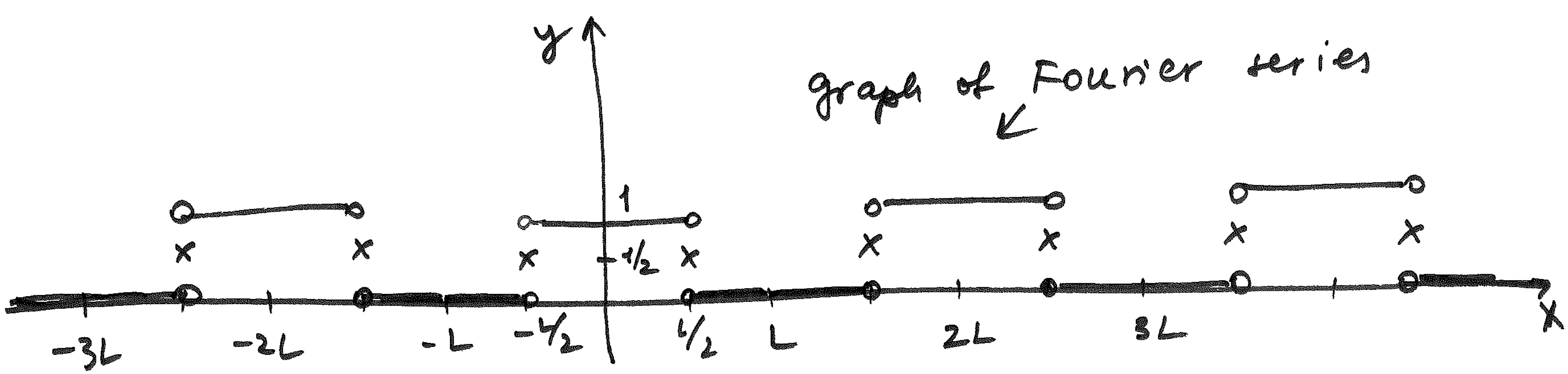
Sketching Fourier series

1. Sketch $f(x)$ on $-L < x < L$.
2. Sketch periodic extension of $f(x)$.
3. Mark an "x" at the average of two limits of $f(x)$ where $f(x)$ has a finite jump discontinuity.

Ex Sketch Fourier series for

$$f(x) = \begin{cases} 1 & |x| < \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$





University of Idaho

$$\therefore\ a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} 1, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & -L \le x < -\frac{L}{2} \\ 0, & \frac{L}{2} < x \le L \\ \frac{1}{2}, & x = \pm\frac{L}{2} \end{cases}$$

Now let's compute Fourier coefficients.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left\{\int_{-L}^{-L/2} \cancelto{0}{f(x)}\,dx + \int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{f(x)}_{=1}\,dx + \int_{\frac{L}{2}}^{L} \cancelto{0}{f(x)}\,dx\right\} = \frac{1}{2L}\cdot L = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{f(x)}_{1}\cos\frac{n\pi x}{L}\,dx =$$

$$= \frac{1}{\cancel{L}} \cdot \frac{\cancel{L}}{n\pi} \sin \frac{n\pi x}{L} \Big|_{x=-\frac{L}{2}}^{x=\frac{L}{2}} = \frac{1}{n\pi} \Big(\sin \frac{n\pi}{L} \cdot \frac{L}{2} - \underbrace{\sin \frac{n\pi \left(-\frac{L}{2}\right)}{L}}_{= -\sin \frac{n\pi \cancel{L}}{2\cancel{L}}} \Big) =$$

$$= \frac{1}{n\pi} \cdot 2 \cdot \sin \frac{n\pi}{2}$$

n odd $\Rightarrow$ $n = 2k+1$ $\Rightarrow$ $\sin \frac{n\pi}{2} = (-1)^k$

n even $\Rightarrow$ $n = 2k$ $\Rightarrow$ $\sin \frac{n\pi}{2} = \sin \frac{2k\pi}{2} = \sin k\pi = 0$

$$\therefore \begin{cases} a_n = 0, & n \text{ is even} \\ a_n = \frac{2}{n\pi} (-1)^k, & n = 2k+1 \quad \text{odd} \end{cases}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_{-L/2}^{L/2} \underbrace{f(x)}_{=1} \cdot \sin \frac{n\pi x}{L} dx =$$

odd over symmetric interval

$$= -\frac{1}{L} \cdot \frac{L}{n\pi} \cos \frac{n\pi x}{L} \Big|_{x=-\frac{L}{2}}^{x=\frac{L}{2}} = 0$$

since cos is even, i.e. $\cos \frac{n\pi \frac{L}{2}}{L} = \cos \frac{n\pi \left(-\frac{L}{2}\right)}{L}$

Hence,

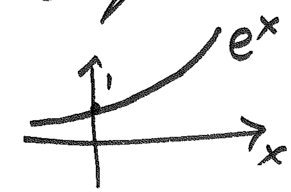
$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} (-1)^k \cos \frac{(2k+1)\pi x}{L}$$

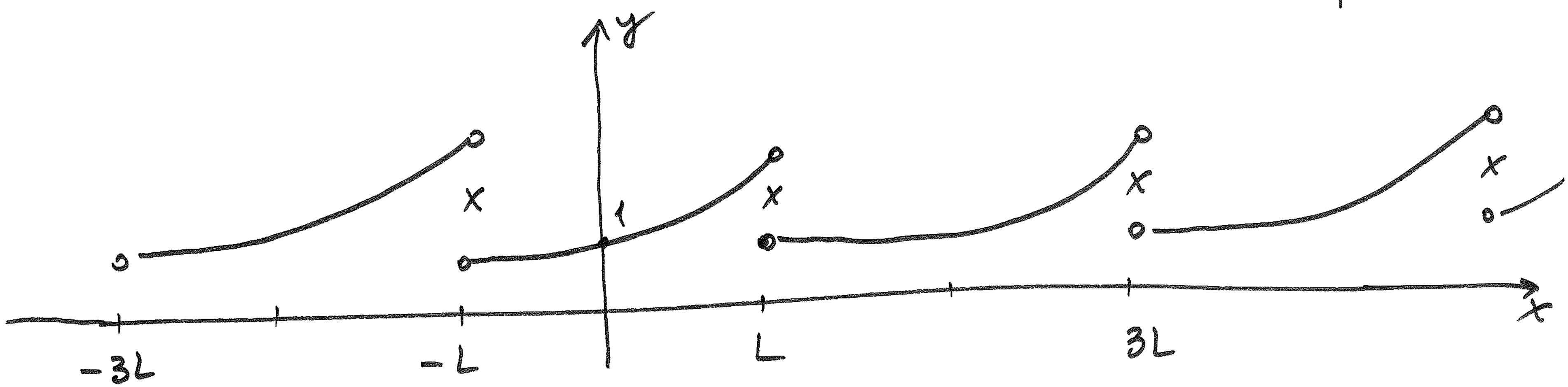
$n = 2k+1$

or

$$f(x) \sim \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L}$$

<u>Ex</u> $f(x) = e^x$. Sketch Fourier series and compute Fourier coefficients.





University of Idaho

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} e^x, & -L < x < L \\ \frac{1}{2}(e^L + e^{-L}), & x = \pm L \end{cases}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} e^x\,dx = \frac{1}{2L} e^x \Big|_{x=-L}^{x=L} = \frac{1}{2L}(e^L - e^{-L})$$