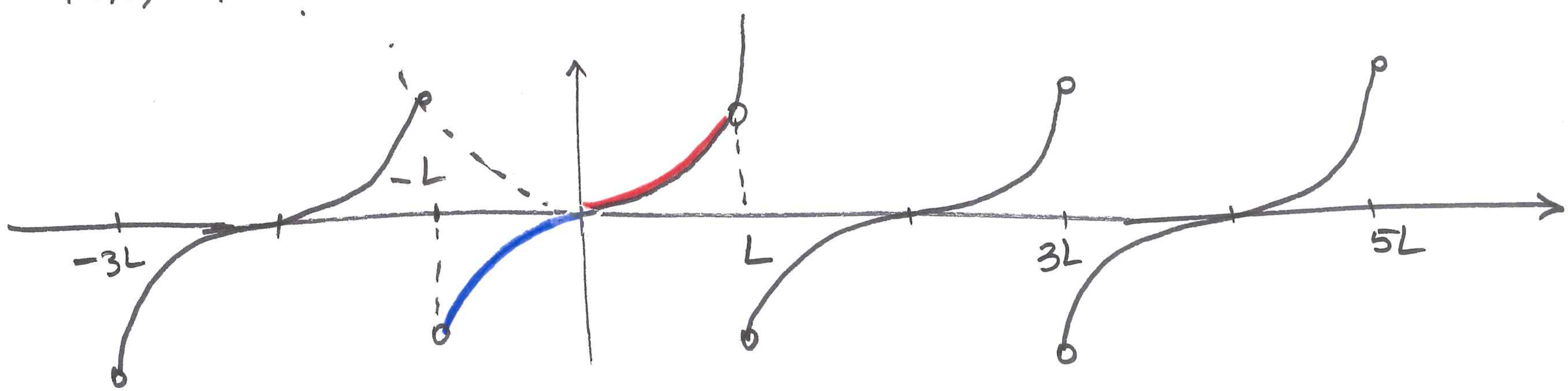
Lecture 17

Ex $f(x) = x^2$. Construct odd extension.



$\therefore$ the odd extension of $f(x)$ has Fourier sine series

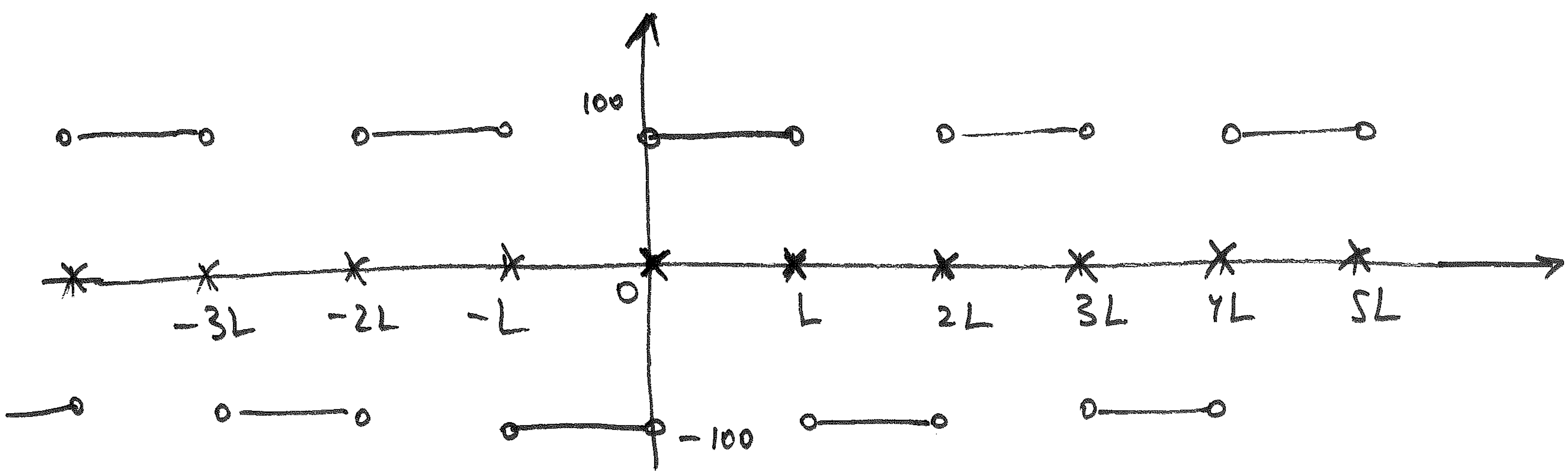
odd extension of $f(x)$ $\sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

Note Fourier convergence Thm (about convergence of a Fourier series) also applies to Fourier sine series.

Sketching Fourier Sine Series

1. Sketch function $f(x)$ on $0 < x < L$.
2. Sketch odd extension of $f(x)$.
3. Mark with "x" the average of values where $f(x)$ has a finite jump discontinuity.

Ex Sketch Fourier sine series for $f(x) = 100$.



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Fourier coefficients

$$B_n = \frac{2}{L}\int_0^L \underbrace{f(x)}_{=100} \sin\frac{n\pi x}{L}\,dx = \frac{200}{\cancel{L}}\cdot\left(-\frac{\cancel{L}}{n\pi}\right)\cos\frac{n\pi x}{L}\Bigg|_{x=0}^{x=L} =$$

$$= -\frac{200}{n\pi}\Big(\underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\cos 0}_{=1}\Big) = -\frac{200}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{100 = f(x) \sim -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\cdot\sin\frac{n\pi x}{L}}$$

Fourier sine series of $f(x)=100$

In the textbook on pages 48-50 (pages 52-54 4^{th} Edition) there is a problem about heat eq^{n} w/ zero Dirichlet BCs:

$$u_t = k\,u_{xx} \qquad 0<x<L, \quad t>0$$

$$u(0,t) = u(L,t) = 0 \qquad t>0$$

$u(x, 0) = 100 \qquad 0 < x < L$

In this example, the initial temperature $f(x) = 100$ is given on $0 < x < L$. But BCs are $u(0,t) = u(L,t) = 0$. As can be seen from the plot of the solution in the textbook, $u(x,t)$ converges to 0 at $x=0$ and $x=L$, hence there is no disagreement w/ BCs.

Please take a look at the example.

Ex Consider again the example with $f(x) = 100$. We note that the odd extension of $f(x)$ is discontinuous at $0, \pm L, \pm 2L, \ldots$

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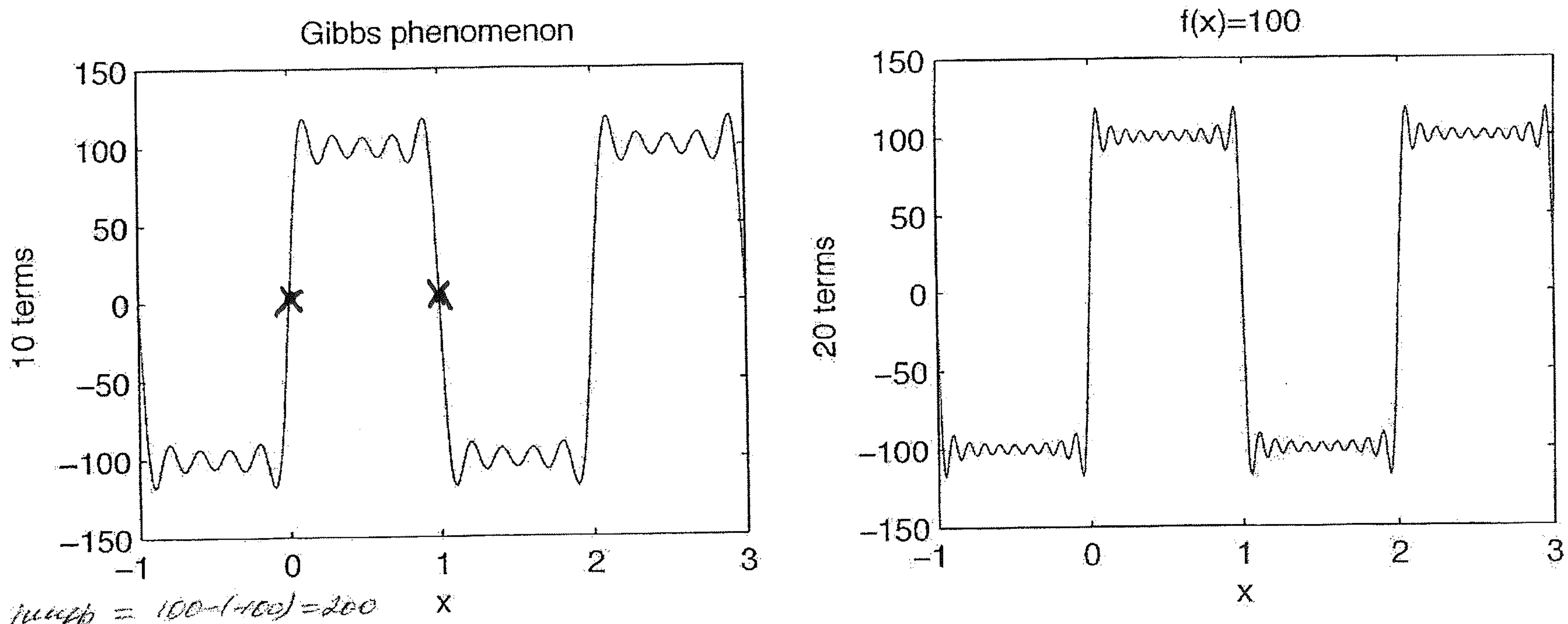
Let $L=1$ and consider only first M terms in the Fourier sine series

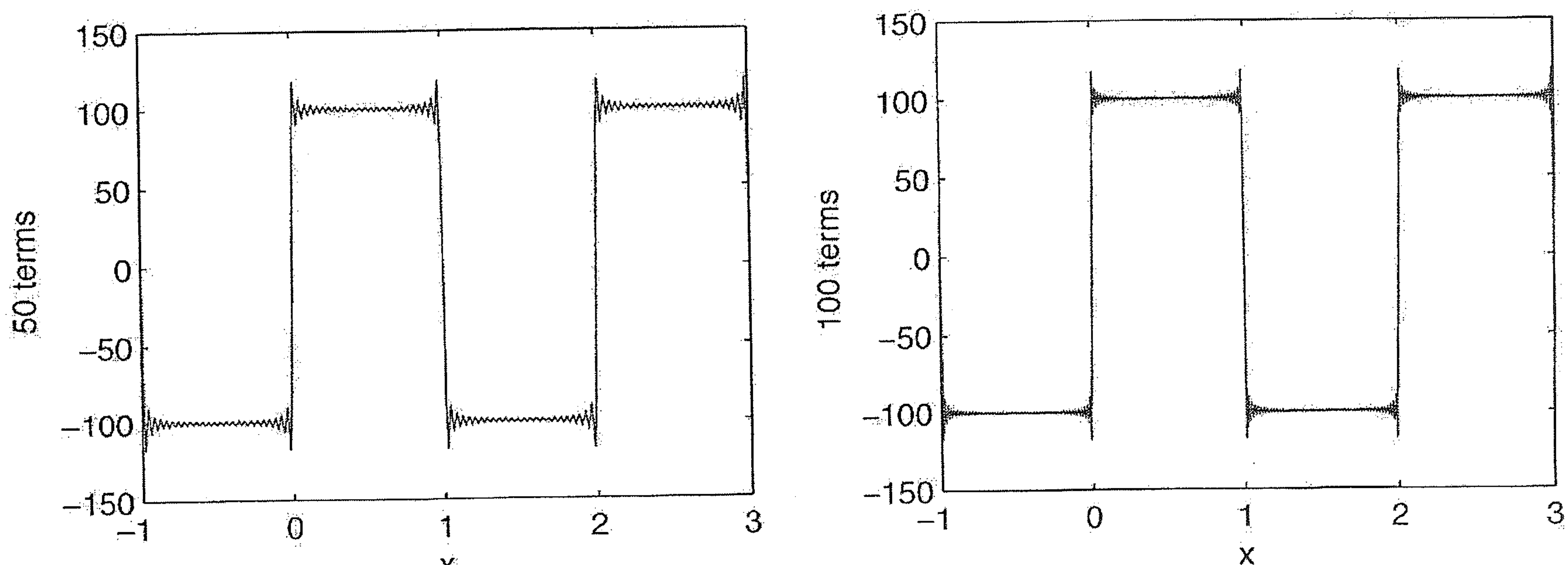
$$f(x) = 100 \sim -\frac{200}{\pi}\sum_{n=1}^{M}\frac{(-1)^n-1}{n}\sin n\pi x \; : \; \text{truncated Fourier series}$$

and consider this finite sum as $M \nearrow$. As $M\to\infty$, the Fourier series converges to 100 in $0<x<L=1$ and goes to 0 at $x=0$ and $x=L=1$. For <u>finite</u> M, oscillations occur near $x=0$ and $x=1$ where the odd extension of $f(x)$ is discontinuous, i.e. Fourier series is discontinuous there.

From the plots of truncated Fourier series w/ $M=10, 20, 50$ and 100, we observe that around points of discontinuity we have oscillations that are called <u>Gibbs phenomenon</u>. These oscillations



Here jump = 100−(−100) = 200
200 × 9% = 18



Overshoot (and undershoot) is ≈ 9% of jump discontinuity have primary overshoot and undershoot at which the finite sum differs from the exact Fourier series by about 9% of the total jump value at the point of discontinuity. Instead of 100 on $(0,1)$, the value of overshoot around $x=0$ is ≈ 118, i.e. the error is $118-100=18$.

Jump at $x=0$ is $[\cdot]\big|_{x=0} = \text{value}(0^+) - \text{value}(0^-)$

$= 100 - (-100) = 200$

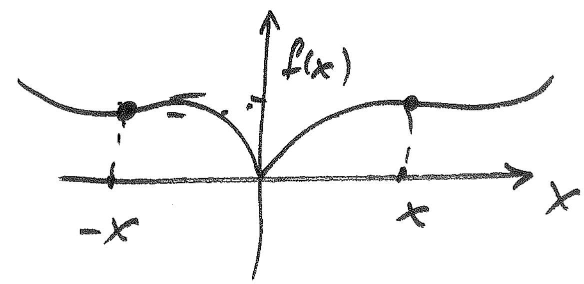
$200 \times 9\% = 200 \times 0.09 = 18$ ✓

Note Amplitude of overshoot / undershoot does <u>NOT</u> decrease as $M \nearrow$, but the truncated Fourier series gives smaller error at points away from the point of discontinuity. Moreover, the width of overshoots and undershoots decreases as $M \nearrow$.

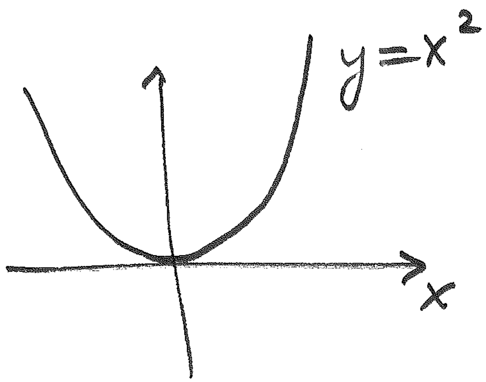
Fourier Cosine Series

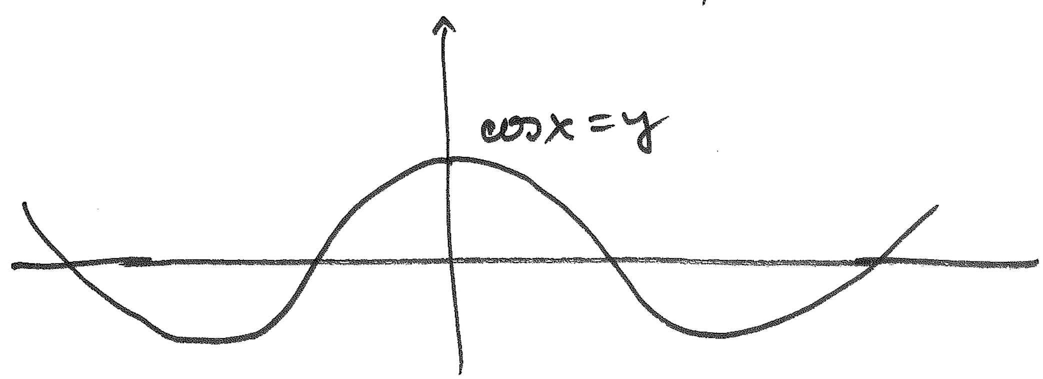
Def An even function $f(x)$ has the property

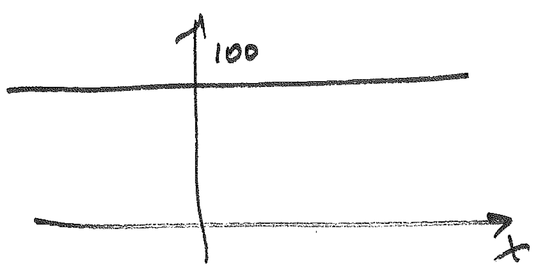
$$f(-x) = f(x)$$

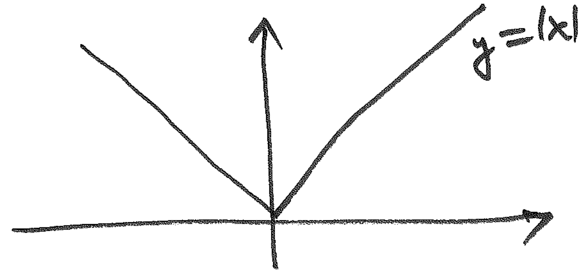


Ex









<u>Note</u> : graph of an even function is symmetric wrt vertical axis (y-axis).