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Lecture 17

Ex $f(x) = e^x$ (Cont'd)

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\underbrace{\int_{-L}^{L} e^x \cos\frac{n\pi x}{L}\,dx}_{=A} \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$= \left| \begin{array}{ll} u = e^x & dv = \cos\frac{n\pi x}{L}\,dx \\ du = e^x\,dx & v = \frac{L}{n\pi}\sin\frac{n\pi x}{L} \end{array} \right| = \frac{1}{L}\left\{ e^x \cdot \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} e^x \sin\frac{n\pi x}{L}\,dx \right\}$$

$$= \left| \begin{array}{ll} u = e^x & dv = \sin\frac{n\pi x}{L}\,dx \\ du = e^x\,dx & v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L} \end{array} \right| = \frac{1}{L}\left\{ \frac{L}{n\pi}e^L \underbrace{\cancel{\sin\frac{n\pi L}{L}}}_{0} - \frac{L}{n\pi}e^{-L}\underbrace{\cancel{\sin\frac{n\pi(-L)}{L}}}_{0} \right.$$

$$\left. - \frac{L}{n\pi}\left[-\frac{L}{n\pi}e^x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} e^x\cos\frac{n\pi x}{L}\,dx \right] \right\} =$$

$$= \frac{1}{L}\left\{ 0 - 0 - \frac{L}{n\pi}\left[-\frac{L}{n\pi}e^L \underbrace{\cos(n\pi)}_{(-1)^n} + \frac{L}{n\pi}e^{-L}\underbrace{\cos(-n\pi)}_{(-1)^n} + \frac{L}{n\pi}A \right] \right\}$$

$$\therefore \; \frac{1}{L} A = -\frac{1}{n\pi}\left[-\frac{L}{n\pi} e^{L}(-1)^n + \frac{L}{n\pi} e^{-L}(-1)^n + \frac{L}{n\pi} A\right]$$

$$\frac{1}{L} A = -\frac{1}{n\pi} \cdot \frac{L}{n\pi}\left[(-1)^n \left(e^{-L} - e^{L}\right) + A\right]$$

$$\frac{1}{L} A = -\frac{L}{(n\pi)^2}\left[(-1)^n \left(e^{-L} - e^{L}\right) + A\right]$$

Solving for A, we get

$$\boxed{A = \frac{L^2}{L^2 + (n\pi)^2} (-1)^n \left(e^{L} - e^{-L}\right)}$$

Then

$$\boxed{a_n = \frac{1}{L} A = \frac{L}{L^2 + (n\pi)^2} (-1)^n \left(e^{L} - e^{-L}\right), \quad n \geq 1} \qquad (1)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{-L}^{L} e^{x} \sin\frac{n\pi x}{L}\, dx$$ (do integration by parts once)

One can show that

$$\boxed{b_n = -\frac{n\pi}{L} \cdot a_n}$$

$\therefore$ Fourier series is

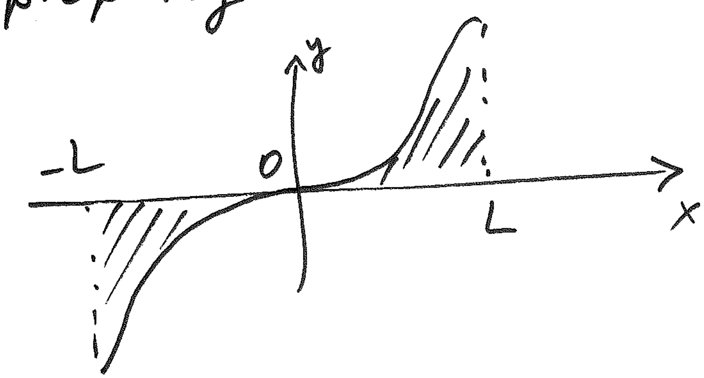
$$\underbrace{\frac{1}{2L}(e^{L}-e^{-L})}_{a_0} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n \sin\frac{n\pi x}{L}$$

where a_n is defined in (1).

Fourier Sine and Fourier Cosine Series

Def An odd function has a property

$$f(-x) = -f(x)$$



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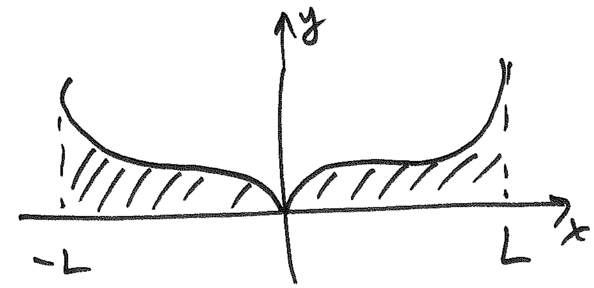
Fourier coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} f(x)\,dx + \frac{1}{2L}\int_{0}^{L} f(x)\,dx =$$

$$\int_{-L}^{0} f(x)\,dx = \left|\begin{array}{l} u=-x \\ du=-dx \\ x=0 \Rightarrow u=0 \\ x=-L \Rightarrow u=L \end{array}\right| = \int_{L}^{0} \underbrace{f(-u)}_{\substack{\shortparallel \\ -f(u) \text{ since } f \text{ is odd}}}(-du) = -\int_{0}^{L} f(u)\,du$$

$$= \frac{1}{2L}\left(-\int_{0}^{L} f(u)\,du\right) + \frac{1}{2L}\int_{0}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}}_{\text{odd}}\,dx = 0, \quad n = 1, 2, \ldots$$



$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\sin\frac{n\pi x}{L}}_{\text{odd}}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$\therefore \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$: <u>Fourier sine series</u>

Recall : solution to $u_t = k u_{xx}$ w/ Dirichlet BCs on $0 < x < L$ is

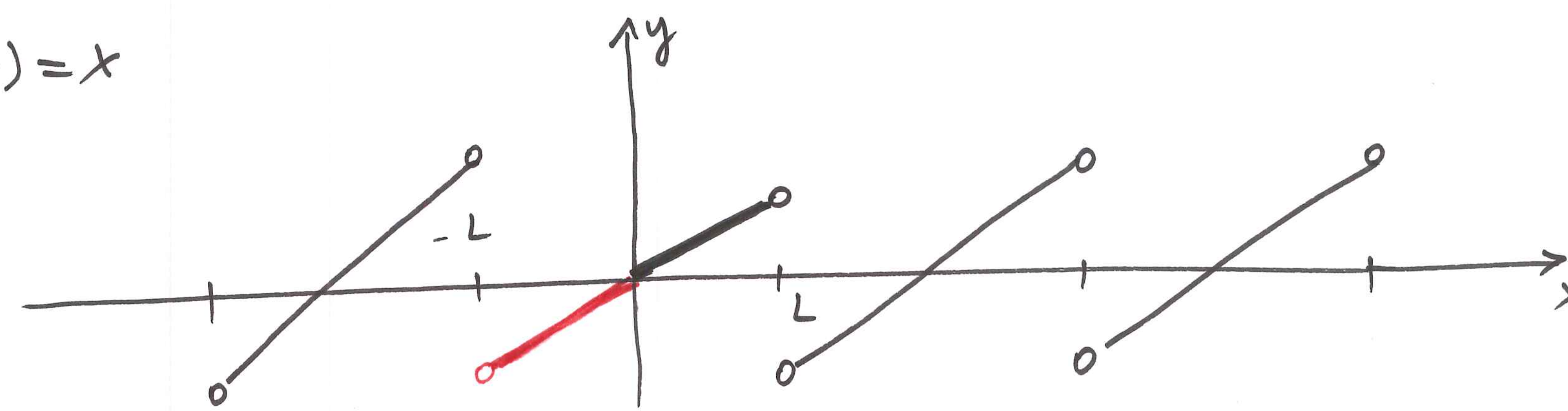
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

IC: $\quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$: Fourier sine series

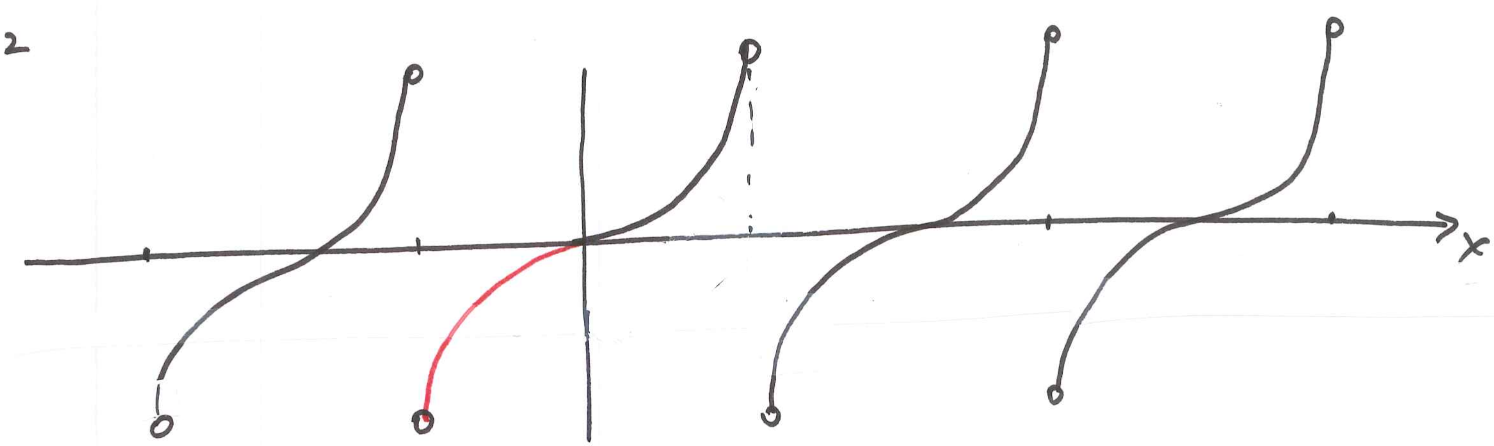
<u>Note</u> The expansion looks the same as Fourier series for an odd function but in the heat equation case function $f(x)$ is <u>ANY</u> function.

Def Let $f(x)$ be a function defined on $0 \le x \le L$. The "odd extension" of $f(x)$ is obtained by "copying", "negating", "flipping" and "pasting" $f(x)$ into intervals $nL \le x \le nL+L$, $n = \pm 1, \pm 2, \ldots$

Ex $f(x) = x$



Ex $f(x) = x^2$



$\therefore$ the odd extension of $f(x)$ has Fourier sine series

$$\text{odd extension of } f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

Note Fourier Thm (about convergence of a Fourier series) also applies to Fourier sine series.

Sketching Fourier Sine Series

1. Sketch function $f(x)$ only on $0 \leq x \leq L$.
2. Sketch odd extension of $f(x)$.
3. Mark with "x" the average of values where $f(x)$ has a finite jump discontinuity.

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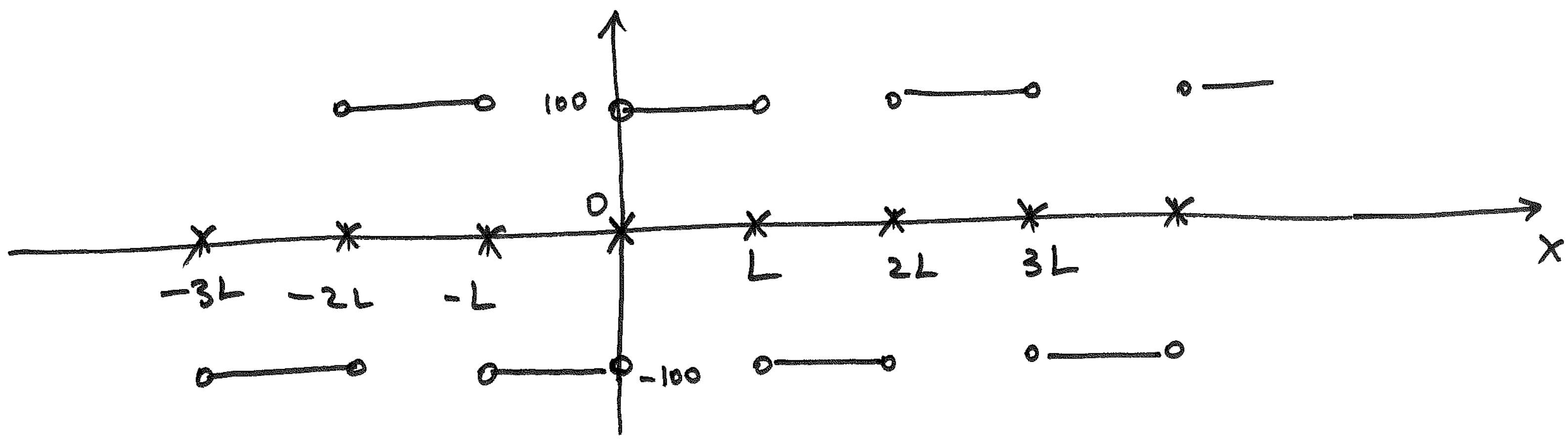
Ex Sketch Fourier sine series for $f(x) = 100$.



Fourier coefficients:

$$B_n = \frac{2}{L}\int_0^L \underbrace{f(x)}_{=100} \sin\frac{n\pi x}{L}\,dx = \frac{2}{\cancel{L}}\cdot 100\left(-\frac{\cancel{L}}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{200}{n\pi}\left[(-1)^n - 1\right]$$

$$\therefore \boxed{100 = f(x) \sim -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\sin\frac{n\pi x}{L}}$$