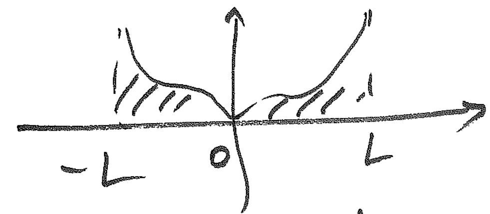
Lecture 18

Fourier Cosine Series (Cont'd)

$f(x)$: even f^n $\qquad f(-x) = f(x)$



Fourier coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even } f^n}\,dx = \frac{1}{2L}\cdot 2\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

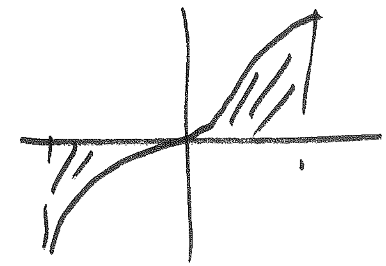
over symmetric interval

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \cos\underbrace{\frac{n\pi x}{L}}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

over symmetric

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{even}}{f(x)}\ \underset{\text{odd}}{\sin\frac{n\pi x}{L}}}_{\text{odd } f^n} \, dx = 0, \qquad n=1,2\ldots$$

over symmetric interval



$$\therefore \quad \boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

Fourier cosine series for even function $f(x)$

Recall: solution to $u_t = k u_{xx}$ on $0 < x < L$ with Neumann BCs, i.e.

$$u_t = k u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u_x(0,t) = u_x(L,t) = 0$$

$$u(x,0) = f(x)$$

has a form

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \quad : \text{ Fourier cosine series}$$

$$\overset{\shortparallel}{f(x)}$$

<u>NOTE</u> This expansion has the same form as a Fourier series for an even function but in the case of the heat equation, $f(x)$ is ANY FUNCTION.

<u>Def</u> Let function $f(x)$ be defined on $0 \le x \le L$. The "even periodic extension" of $f(x)$ is obtained by "copying", "flipping" and "pasting" $f(x)$ into

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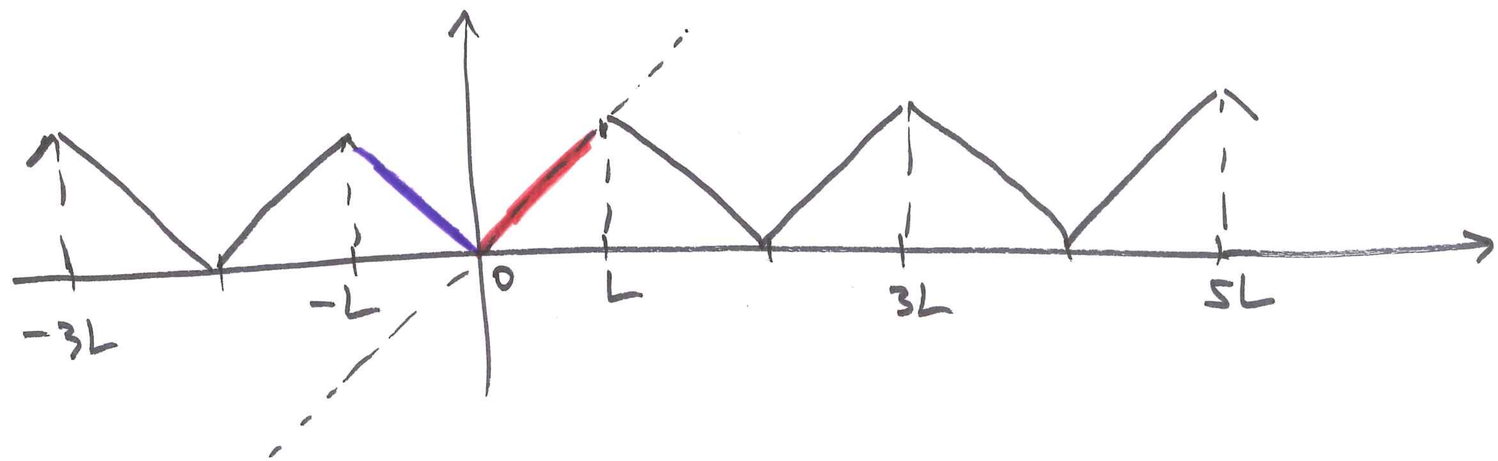
subintervals $nL \le x \le nL + L$, $n = \pm 1, \pm 2, \ldots$

Ex $f(x) = x$. Sketch even periodic extension.



Note: even periodic extension is automatically continuous at $x=0, \pm L$ (It is continuous by construction).

even periodic extension of $f(x)$ $\sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$: Fourier cosine series where $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Fourier convergence Thm also applies to Fourier cosine series.

<u>How to sketch Fourier cosine series</u>

1. Sketch function $f(x)$ on $0 \le x \le L$.
2. Sketch the even periodic extension of $f(x)$.
3. Mark with "x" the average of values at the points where the even periodic extension has finite jump discontinuity.

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3.3.4 The even and odd parts of a function

Q Are the Fourier series, Fourier sine series and Fourier cosine series related? Yes.

Claim ANY function $f(x)$ can be written as a sum of an even and an odd functions.

Pf

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x) \right]}_{\substack{f_e(x) \\ \text{even part} \\ \text{of } f(x)}} + \underbrace{\frac{1}{2}\left[f(x) - f(-x) \right]}_{\substack{f_o(x) \\ \text{odd part} \\ \text{of } f(x)}}$$

$$f_e(-x) = \frac{1}{2}\left[f(-x) + f(x) \right] = f_e(x)$$

$x \to -x$

$\therefore$ $f_e(x)$ is an even $f^{\underline{n}}$

$$\underset{x \to -x}{f_o(-x) =} \frac{1}{2}\left[f(-x) - f(-(-x)) \right] = \frac{1}{2}\left[f(-x) - f(x) \right] =$$

$$= -f_o(x)$$

$\therefore$ $f_o(x)$ is an odd $f^{\underline{n}}$

<u>Ex</u> $\quad f(x) = \dfrac{1}{x+1}$

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x) \right] = \frac{1}{2}\left[\frac{1}{x+1}^{(-x+1} + \frac{1}{-x+1}^{(x+1} \right] =$$

$$= \frac{1}{2}\,\frac{-\cancel{x}+1+\cancel{x}+1}{(x+1)(1-x)} = \frac{1}{2}\,\frac{2}{1-x^2} = \frac{1}{1-x^2} = f_e(x):$$

even part of $f(x)$

$$(a-b)(a+b) = a^2-b^2$$

$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right] = \frac{1}{2}\left[\frac{1}{x+1}^{(-x+1} - \frac{1}{-x+1}^{(x+1}\right]$$

$$= \frac{1}{2}\,\frac{-x+1-x-1}{1-x^2} = \frac{1}{2}\,\frac{(-2x)}{1-x^2} = \frac{x}{x^2+1} = f_o(x):$$

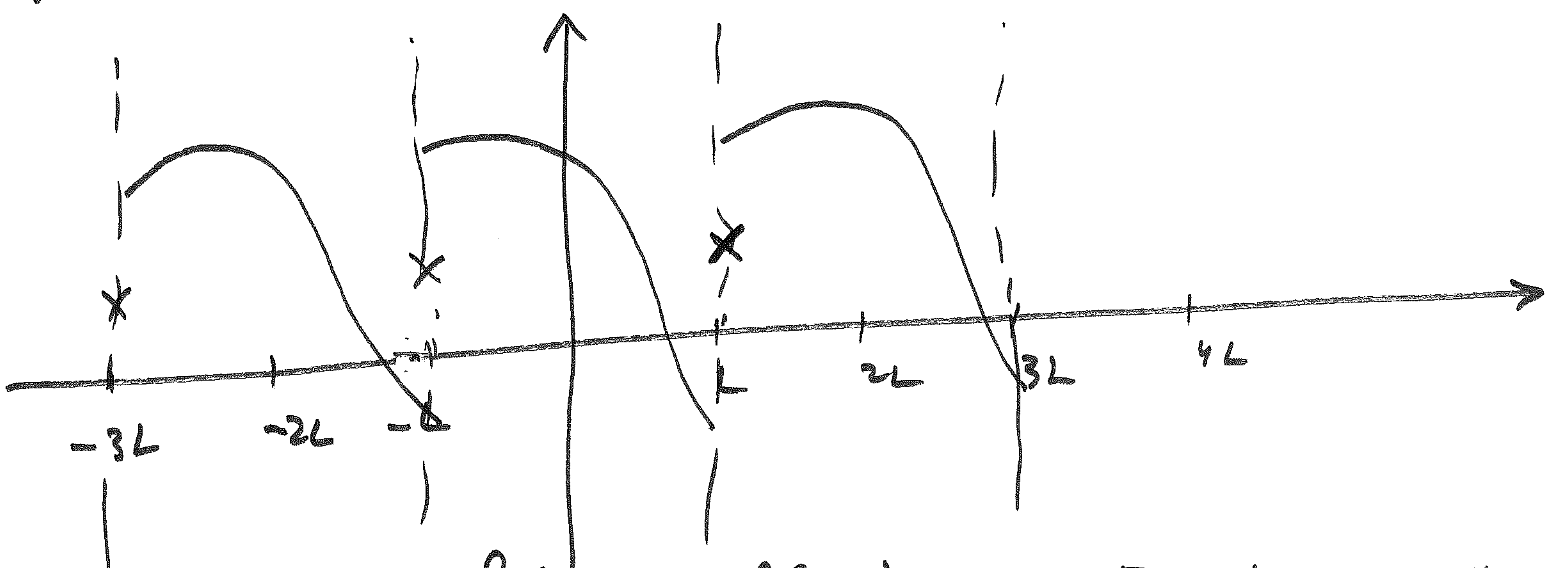
odd part of $f(x)$

<u>Thm</u> Fourier series of $f(x)$ equals Fourier cosine series of an even part of $f(x)$ plus Fourier sine series of an odd part of $f(x)$.

3.3.5 Convergence of Fourier series

Assume that $f'(x)$ is piecewise smooth.

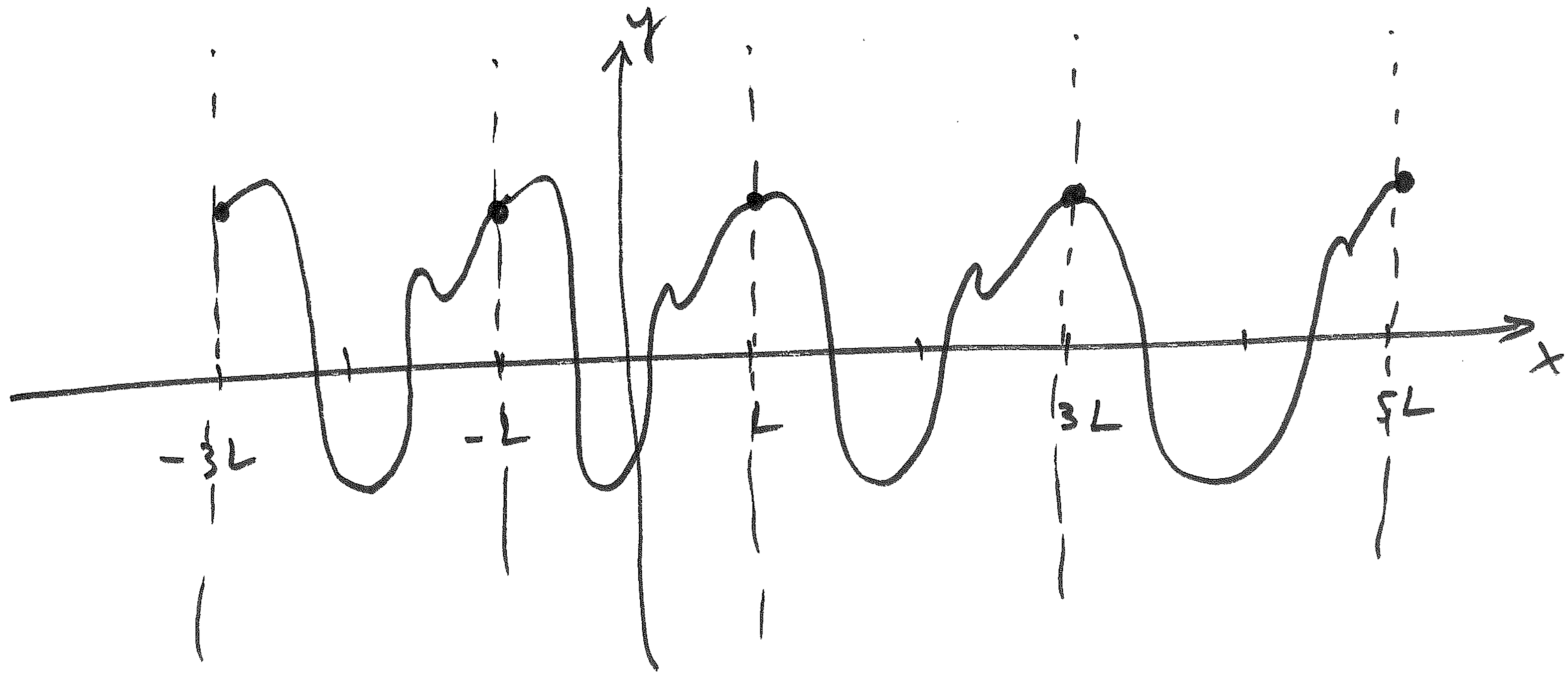
Thm 1 The Fourier series of $f(x)$ is continuous and converges to $f(x)$ on $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.



We can see that $f(L) \neq f(-L) \Rightarrow$ Fourier series is NOT continuous at $x = (2n+1)L$.

$\therefore$ Fourier series does <u>not</u> converge to $f(x)$ at $x = (2n+1)L$.

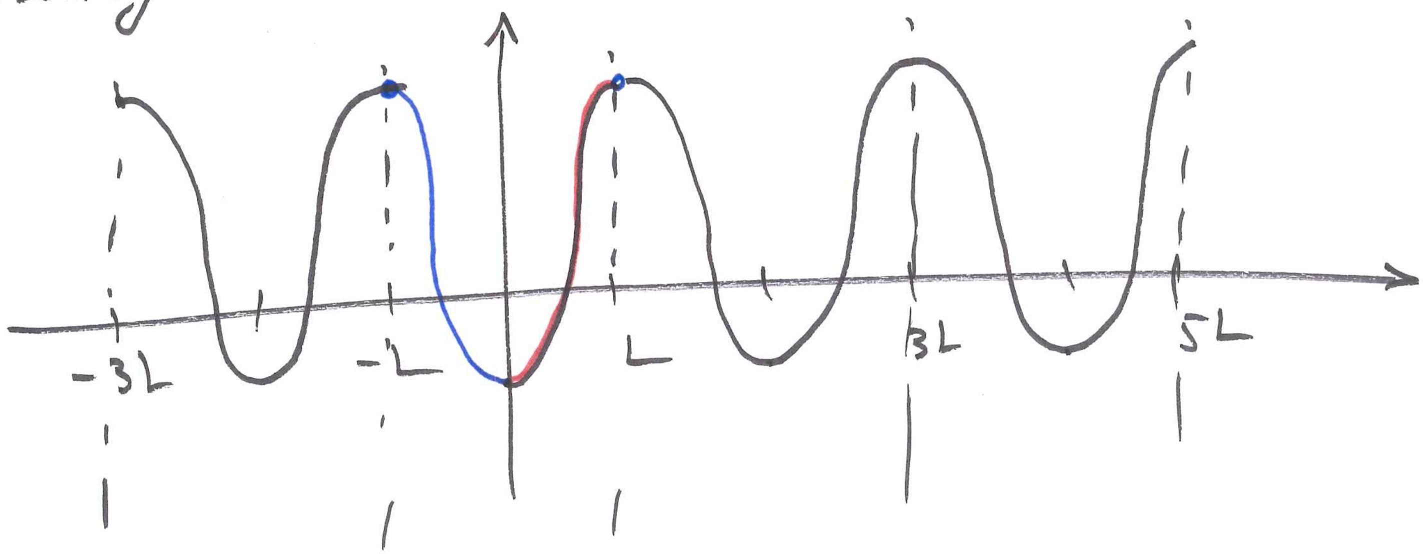
<u>Ex</u>



$f(-L) = f(L) \Rightarrow$ Fourier series is continuous $\Rightarrow$ Fourier series converges to $f(x)$ on $-L \le x \le L$

Thm 2 The Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ on $-L \leq x \leq L$ if and only if $f(x)$ is continuous.

Note that the even extension guarantees continuity at $x=0$ and $x=L$.



Even extension of $f(x)$ is continuous at $x=0$ and $x=L$ by construction.