Lecture 18

In the textbook on pages 48-50 there is a problem

$$u_t = k u_{xx} \qquad 0 < x < L, \quad t > 0$$

$$u(0,t) = u(L,t) = 0 \qquad t > 0$$

$$u(x,0) = 100 \qquad 0 < x < L$$

i.e. heat flow w/ Dirichlet BCs. In this example, the initial temperature $f(x) = 100$ is given on $0 < x < L$ but BCs are $u(0,t) = u(L,t) = 0$. Solution converges to 0 at $x = 0$ and $x = L$, hence, there is no disagreement w/ BCs.

Please take a look at this example.

__Ex__ Consider again example with $f(x) = 100$. The odd extension of $f(x)$ is discontinuous at $0, \pm L, \pm 2L, \ldots$

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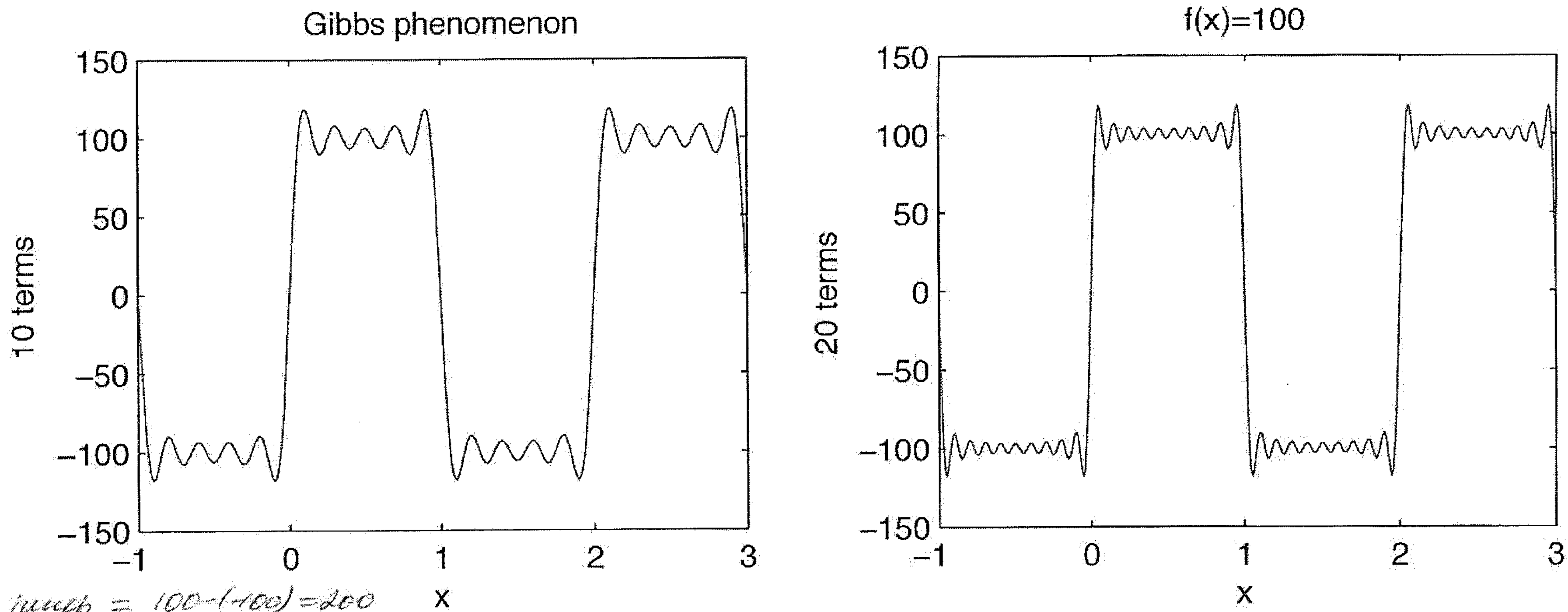
If we set $L=1$, then

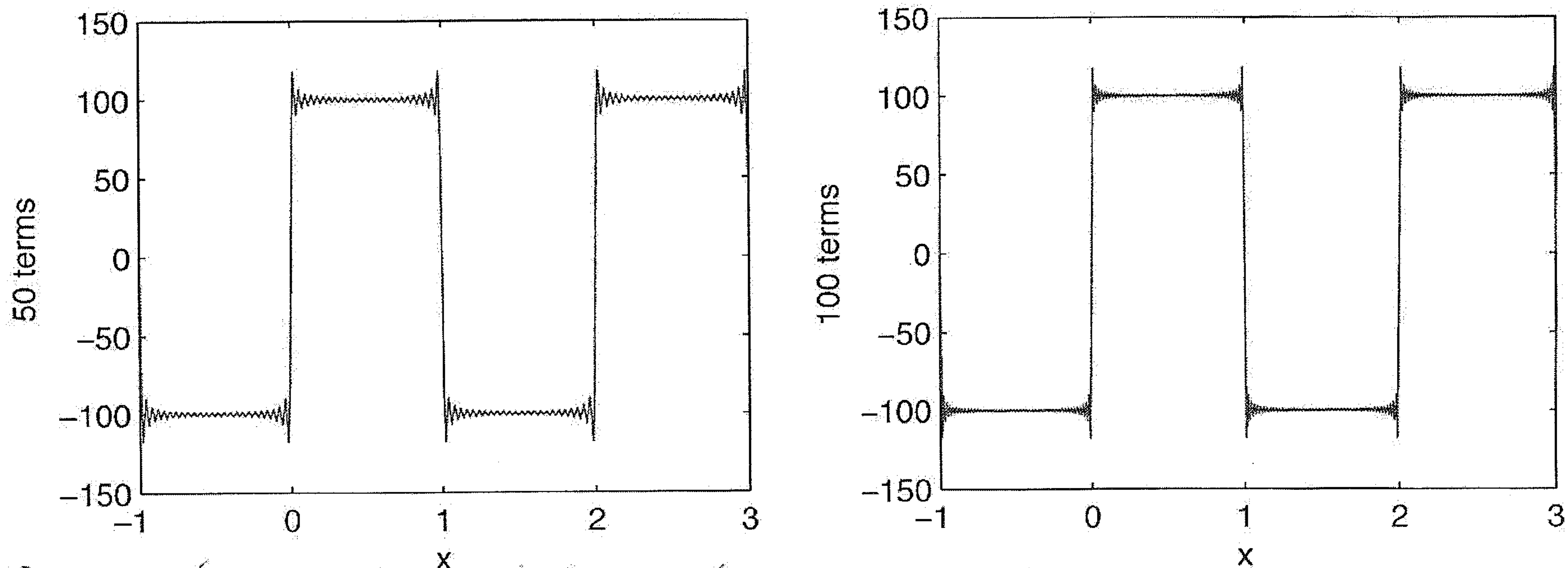
$$100 \approx \sum_{n=1}^{M} \frac{200}{n\pi}\left[1-(-1)^n\right]\sin n\pi x$$

and as $M \to \infty$, the Fourier series converges to 100 in $0 < x < L = 1$ and goes to 0 at $x=0$ and $x=L=1$.
For finite M, oscillations occur at $x=0$ and $x=1$ where the odd extension of $f(x)$ is discontinuous (i.e. where the Fourier series is discontinuous).

From the plots of truncated Fourier sine series w/ $M = 10, 20, 50$ and 100, we observed that at the points of discontinuity we have so called Gibbs phenomenon with primary overshoot (and corresponding undershoot) at which finite sum differs from the exact Fourier series by about 9% of the total jump at the point of discontinuity, i.e. instead of 100,



Here jump = 100−(−100) = 200
200 × 9% = 18



Overshoot (and undershoot) is ≈ 9% of jump discontinuity the value at the overshoot is ≈ 118, i.e. the error is ≈ 18.

Jump at $x=0$:

$$[\ldots]_{x=0} = \text{value}(x=0^+) - \text{value}(x=0^-) = 100 - (-100) = 200$$

$$200 \cdot 9\% = 18$$

Note Amplitude of overshoots / undershoots does not decrease as $M \nearrow$, but the truncated Fourier series gives smaller error at points away from points of discontinuity. Moreover, the width of overshoots and undershoots decreases as $M \nearrow$.

Fourier Cosine Series

Def An even function $f(x)$ has the property $f(-x) = f(x)$.

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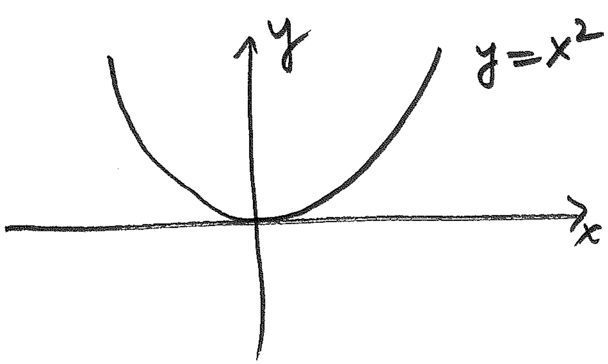

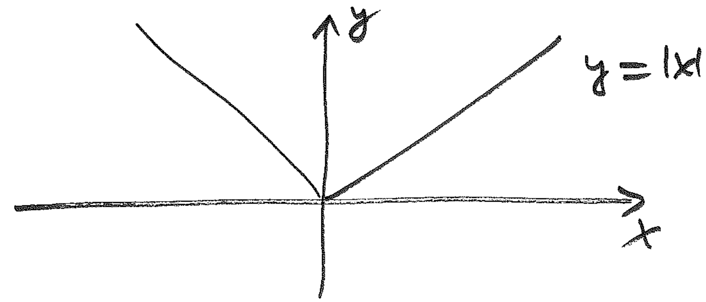

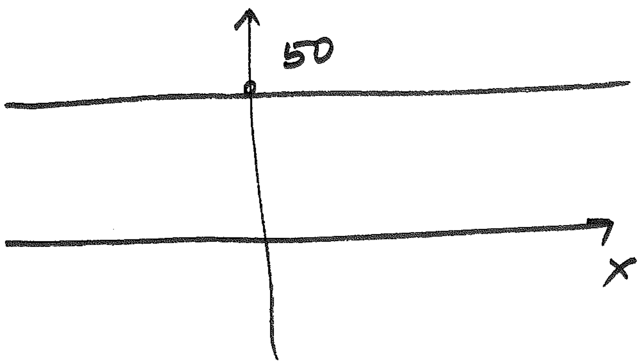

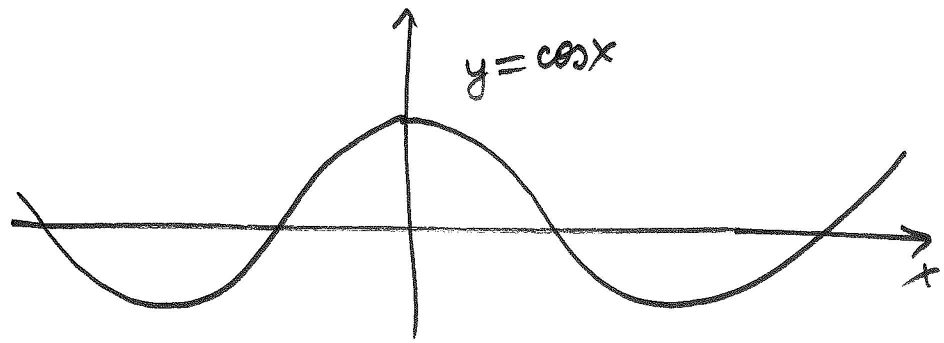

Graph of an *even* function is symmetric wrt y-axis.

Fourier coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\,dx = \frac{1}{2L}\cdot 2\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{even}}{f(x)}\,\underset{\text{even}}{\cos\frac{n\pi x}{L}}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{even}}{f(x)}\,\underset{\text{odd}}{\sin\frac{n\pi x}{L}}}_{\text{odd}}\,dx = 0$$

$$\therefore \quad f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} \; :$$

<u>Fourier cosine series</u>

Recall: solution to $u_t = k u_{xx}$ on $0 < x < L$ with Neumann BCs has the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \ :$$ Fourier cosine series

NOTE This has the same form as a Fourier series for an even function but in the heat problem function $f(x)$ is ANY function.

Def Let function $f(x)$ be defined on $0 \le x \le L$. The even periodic extension of $f(x)$ is obtained by "copying", "flipping" and "pasting" $f(x)$ into intervals $nL \le x \le nL + L$, $n = \pm 1, \pm 2, \ldots$

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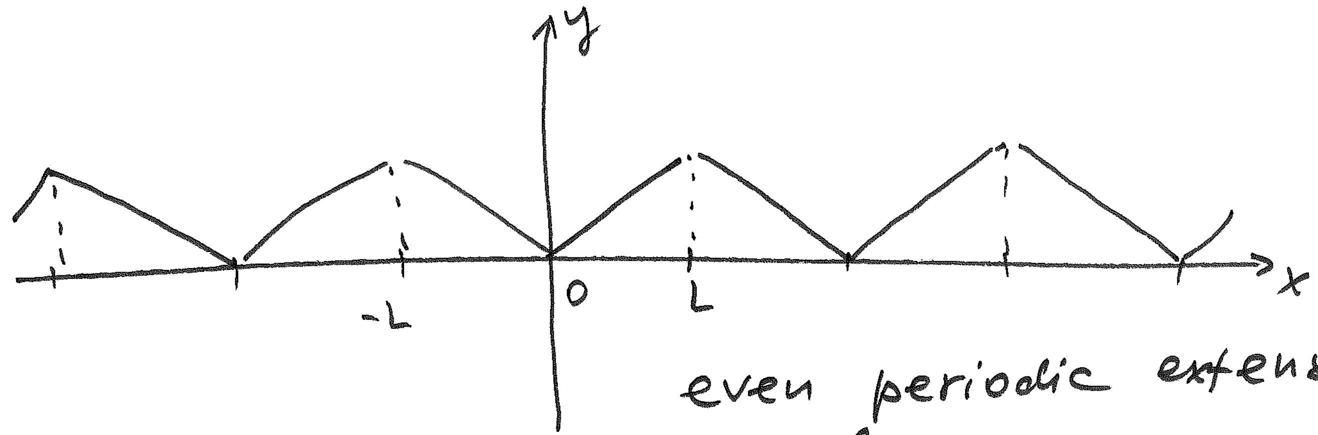
Ex $f(x) = x$



even periodic extension of $f(x)=x$

Note: even periodic extension is automatically continuous at $x = 0, \pm L$ (or continuous by construction).

The even periodic extension $\sim \displaystyle\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$: Fourier cosine series

where $A_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Fourier Thm also applies to Fourier cosine series.

How to sketch Fourier cosine series

1. Sketch function $f(x)$ only on $0 \leq x \leq L$.
2. Sketch the even periodic extension of $f(x)$.
3. Mark with "x" the average of values at the points of discontinuity of the extension.