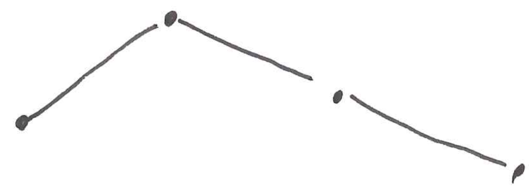
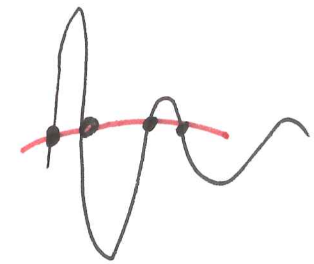
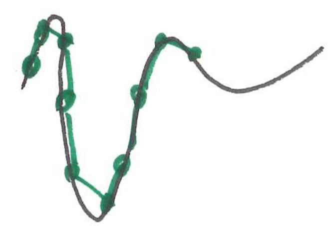
Lecture 19





$x = 0 : 1e^{-1} : 1$

$\Delta x = 0.1$

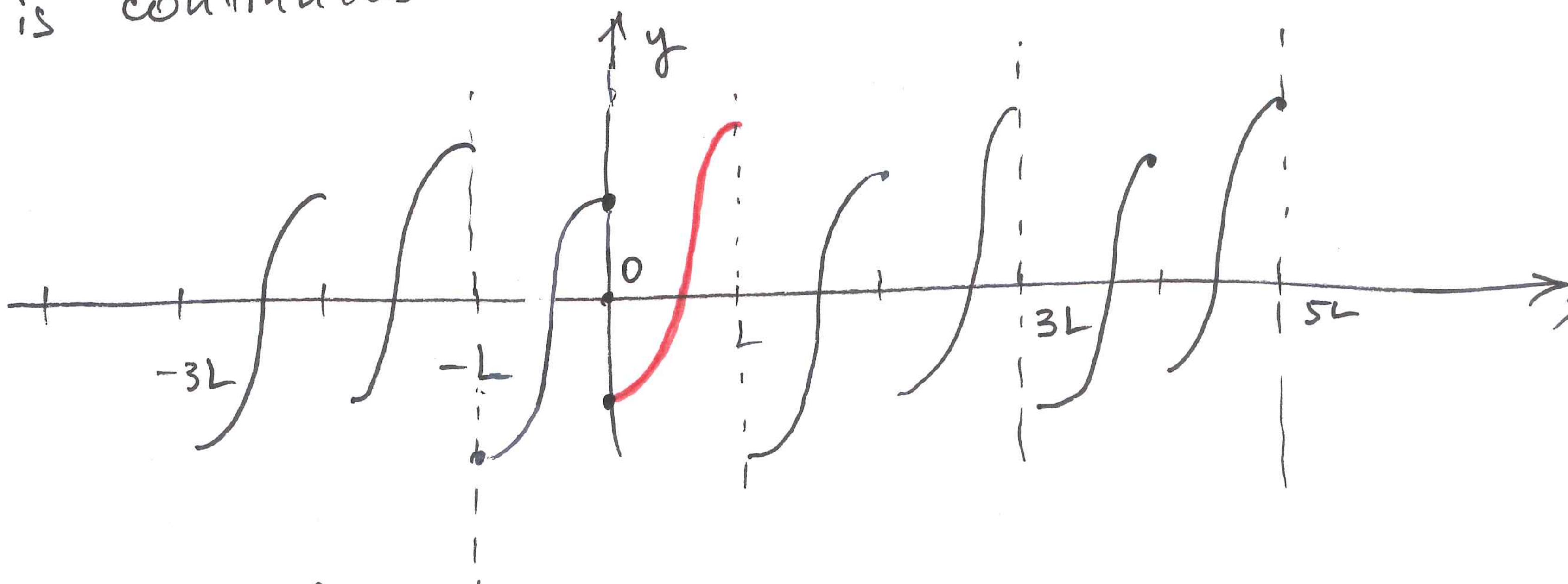


$x = 0 : 1e{-3} : 1$

Note: when you plot a truncated Fourier series, make sure you have enough resolution (i.e. # of points) to resolve/see Gibbs phenomenon.

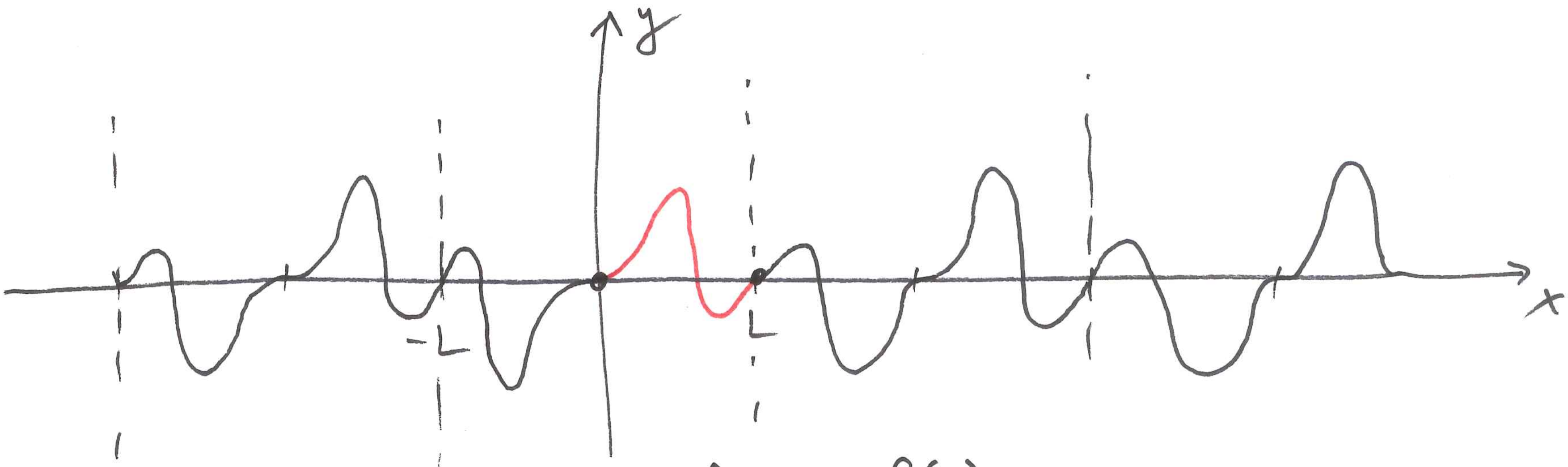
Convergence of Fourier series (Cont'd)

Thm 3 The Fourier sine series of $f(x)$ is <u>continuous</u> and <u>converges</u> to $f(x)$ on $0 \le x \le L$ if and only if $f(x)$ is continuous and $f(0) = f(L) = 0$.



$f(0) \neq 0$, $f(L) \neq 0$

Hence, the odd periodic extension and, hence, Fourier sine series are <u>NOT</u> continuous at $x=0$ and $x=L$.



$f(0) = f(L) = 0$

Hence, the odd periodic extension and Fourier sine series are continuous at $x=0$ & $x=L$.

3.4 Term-by-term differentiation of Fourier series

Recall

$$u_t = k u_{xx} \qquad 0 < x < L$$

$$u(0,t) = u(L,t) = 0$$

University of Idaho

$$u(x,0) = f(x)$$

By separation of variables and using the principle of linear superposition, we found solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Q Does this function really satisfy the heat eq^n?

Assume ~~that~~ this series can be differentiated term-by-term.

$$u_t = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \left(-k\left(\frac{n\pi}{L}\right)^2\right) \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u_{xx} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \left(-\left(\frac{n\pi}{L}\right)^2\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$\therefore \quad u_t = k u_{xx}$

Q Can we differentiate term-by-term in general?

Ex Consider Fourier sine series for $f(x) = x$

$$x \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L}\int_0^L x \cdot \sin\frac{n\pi x}{L}\,dx \underset{\text{parts}}{\overset{\text{by}}{=}} \frac{2L}{n\pi}(-1)^{n+1}$$

$$\Rightarrow \quad x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \cdot \sin\frac{n\pi x}{L} \qquad (*)$$

Let's differentiate $(*)$ formally wrt x.

$$1 \overset{?}{\sim} \sum_{n=1}^{\infty} \frac{2\cancel{L}}{\cancel{n\pi}}(-1)^{n+1} \cdot \cancel{\frac{n\pi}{L}} \cos\frac{n\pi x}{L} = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos\frac{n\pi x}{L} \qquad (**)$$

But Fourier series of 1 is just 1:

$$1 = \underset{\substack{\parallel \\ a_0}}{1} + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{n\pi x}{L} \qquad a_n = 0,\ n \geq 1,\ a_0 = 1$$

Therefore, series in $(**)$ is not a Fourier series of function $1 = f'(x)$.

Moreover, n^{th} term $2(-1)^{n+1} \cos \frac{n\pi x}{L} \not\to 0$ as $n \to \infty$ (necessary condition for a convergence of a series). Hence, Fourier series on the RHS of $(**)$ does not even converge.

$\therefore$ Not every Fourier series can be differentiated term-by-term.

Theorem

a Fourier series that is <u>continuous</u> (has no jump discontinuities) can be differentiated term-by-term if $f'(x)$ is piecewise smooth.

alternatively, we can write explicitly what continuity of Fourier series means.

<u>Thm 4</u> If $f'(x)$ is piecewise smooth, then Fourier series of a continuous function $f(x)$ can be differentiated term-by-term if $f(-L) = f(L)$.

Thm 5 If $f'(x)$ is piecewise smooth, then Fourier series of a continuous function $f(x)$ can always be differentiated term-by-term.

Thm 6 If $f'(x)$ is piecewise smooth, then Fourier series of a continuous function $f(x)$ can be differentiated term-by-term if $f(0) = f(L) = 0$.

Note Jump discontinuities in the domain or at endpoints are the reason that causes term-by-term differentiation to be invalid.