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Lecture 19

The even and odd parts of a function

Q Are the Fourier series, Fourier sine series and Fourier cosine series related? Yes.

Claim ANY function $f(x)$ can be written as a sum of an even and an odd functions.

Pf Let

$$f(x) = \underbrace{\tfrac{1}{2}\left[f(x) + f(-x)\right]}_{\substack{f_e(x) \\ \text{even part} \\ \text{of } f(x)}} + \underbrace{\tfrac{1}{2}\left[f(x) - f(-x)\right]}_{\substack{f_o(x) \\ \text{odd part} \\ \text{of } f(x)}}$$

$$f_e(-x) = \frac{1}{2}\left[f(-x) + f(\underbrace{-(-x)}_{x})\right] = f_e(x)$$

$\therefore$ $f_e(x)$ is even function

$$f_o(-x) = \frac{1}{2}\left[f(-x) - f(-(-x))\right] = \frac{1}{2}\left[f(-x) - f(x)\right] = -f_o(x)$$

$\therefore$ $f_o(x)$ is odd function

<u>Ex</u> $f(x) = \frac{1}{x+1}$

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right] = \frac{1}{2}\left[\frac{1}{x+1}^{(-x+1} + \frac{1}{-x+1}^{(x+1}\right] = \frac{1}{2}\,\frac{\cancel{-x}+1+\cancel{x}+1}{(x+1)(-x+1)} =$$

$= \frac{1}{1-x^2}$: even part of $f(x)$

$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right] = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{-x+1}\right] = \frac{1}{2}\,\frac{-x+1-(x+1)}{(x+1)(-x+1)} =$$

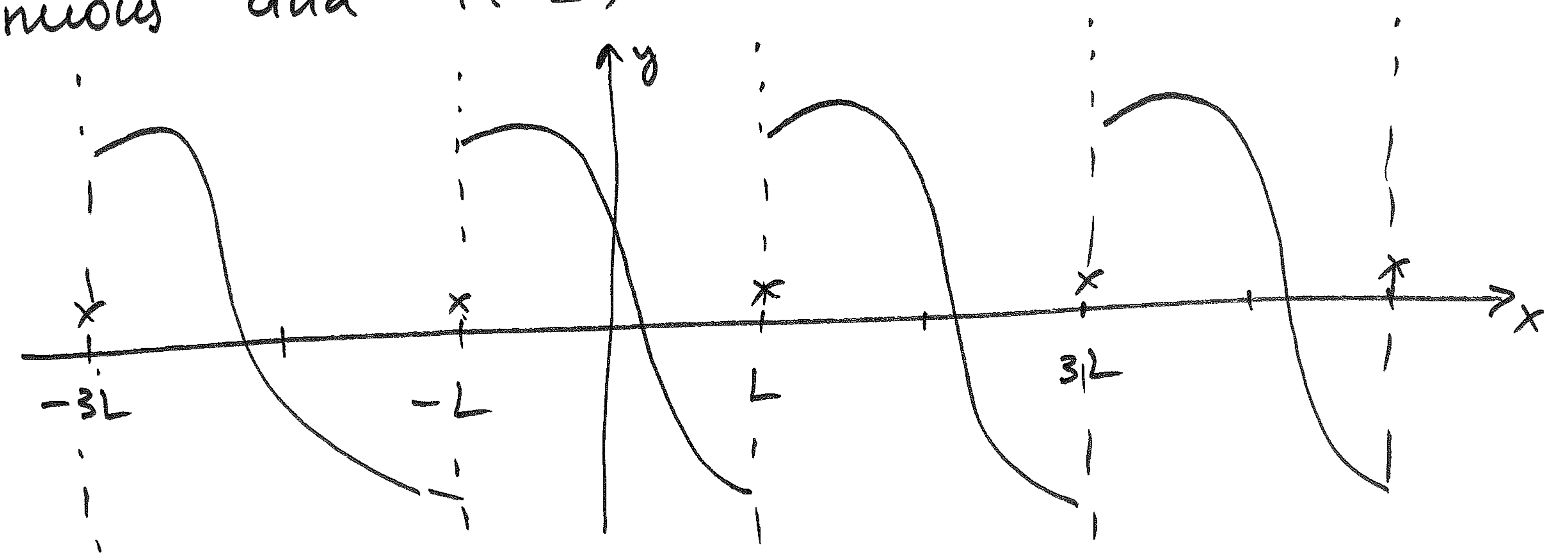
$= -\frac{x}{1-x^2} = \frac{x}{x^2-1}$: odd part of $f(x)$

<u>Thm</u> Fourier series of $f(x)$ equals Fourier cosine series of an even part $f_e(x)$ of $f(x)$ plus Fourier sine series of an odd part $f_o(x)$ of $f(x)$.

Convergence of Fourier series

Assume that $f'(x)$ is piecewise smooth.

<u>Thm 1</u> The Fourier series of $f(x)$ is <u>continuous</u> and <u>converges</u> to $f(x)$ for $-L \leq x \leq L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.



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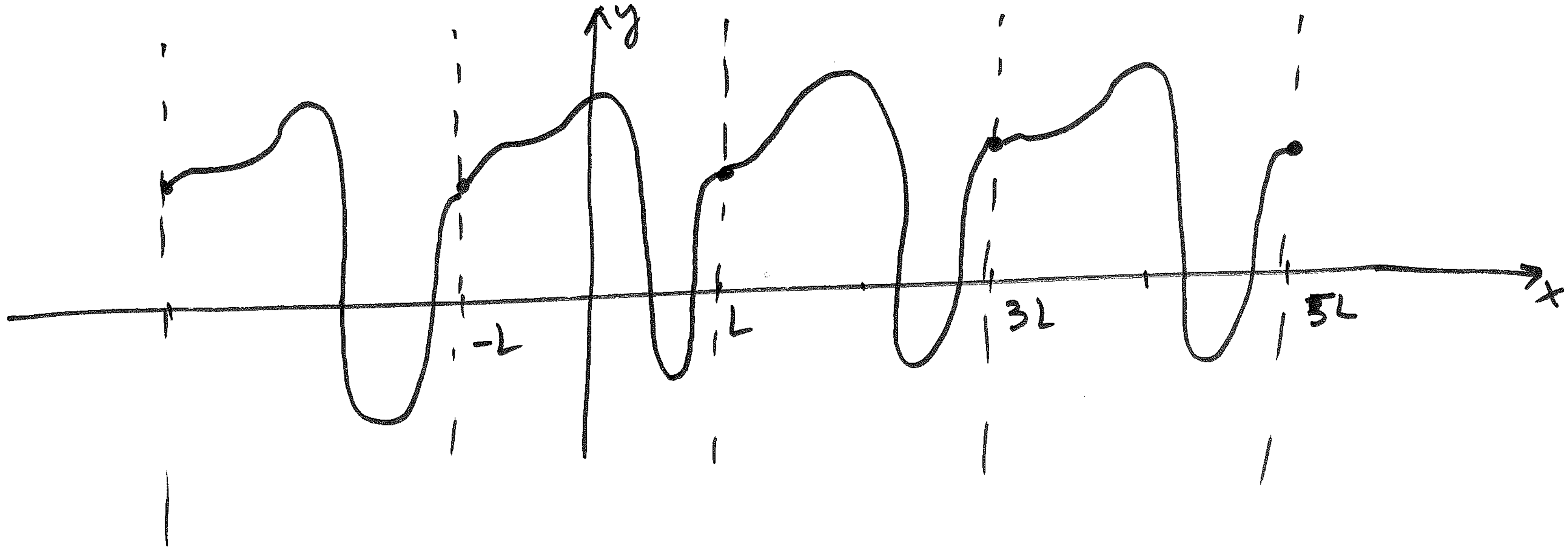
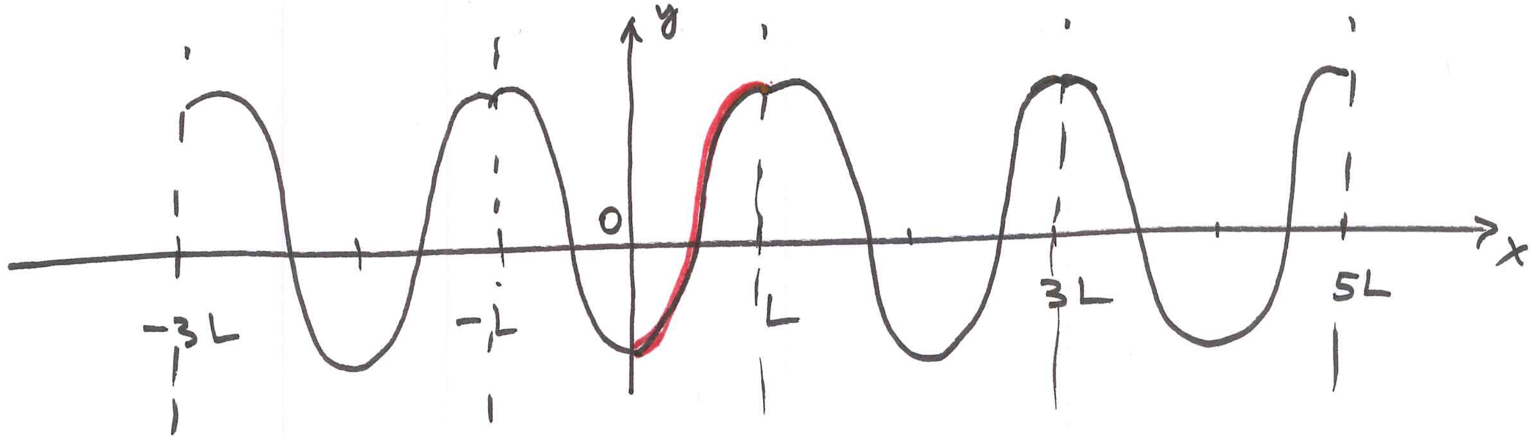
$f(-L) \neq f(L) \Rightarrow$ Fourier series is **not** continuous at $(2n+1)L$

$\therefore$ Fourier series doesn't converge to $f(x)$ at $x = (2n+1)L$



$f(-L) = f(L) \Rightarrow$ Fourier series is continuous $\Rightarrow$ Fourier series converges to $f(x)$ on $-L \leq x \leq L$.

Thm 2 The Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous. Note that the even extension guarantees continuity at $x=0$ and $x=L$.



Even extension of $f(x)$ is continuous at $x=0$ and $x=L$ by construction.

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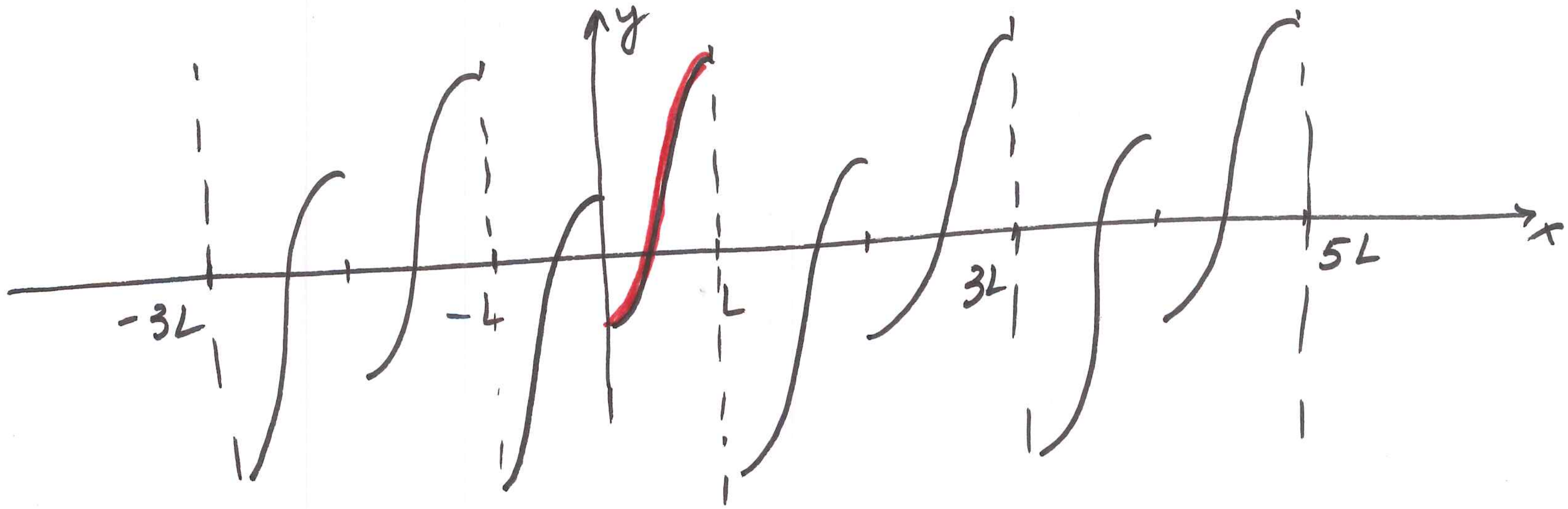
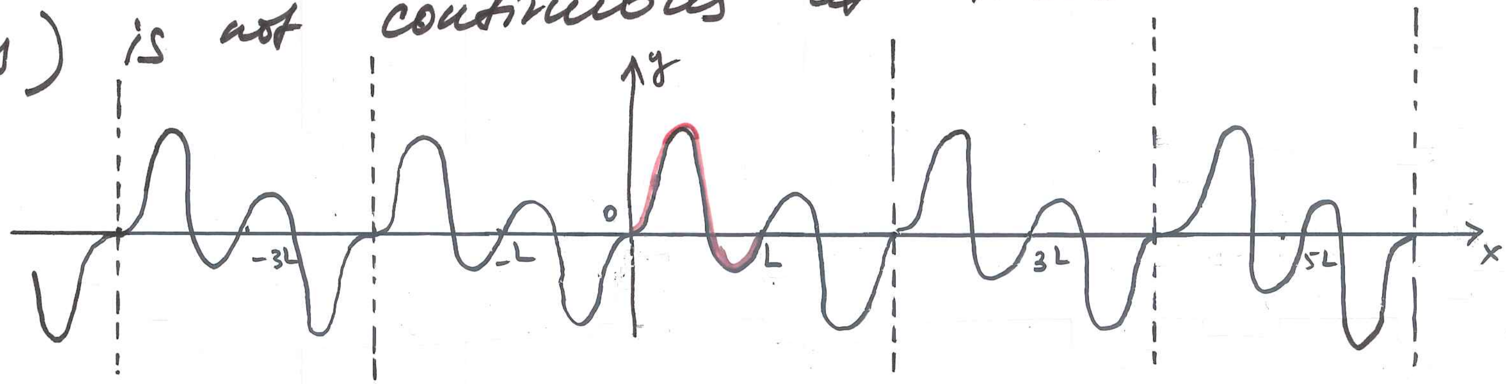
Thm 3 The Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ on $0 \le x \le L$ if and only if $f(x)$ is continuous and $f(0) = f(L) = 0$.



$f(0) \neq 0$, $f(L) \neq 0$

Here the odd extension (and hence Fourier sine series) is not continuous at $x=0$ and $x=L$.



$f(0)=f(L)=0$

$\Downarrow$

Fourier sine series is continuous

Term-by-term differentiation of Fourier series

Recall

$$u_t = k u_{xx} \qquad 0 < x < L$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$

By separation of variables and using the principle of linear superposition we obtained the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Q Does it really satisfy the heat equation?

Assume that this series can be differentiated term-by-term:

$$u_t = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\left(-k\left(\frac{n\pi}{L}\right)^2\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u_{xx} = \sum_{n=1}^{\infty} B_n \left(-\left(\frac{n\pi}{L}\right)^2\right) \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\therefore \quad u_t = k\, u_{xx}$$

<u>Q</u> Can we differentiate term-by-term in general?

<u>Ex</u> Consider Fourier sine series for $f(x) = x$.

$$x \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L}\int_0^L \underbrace{x}_{f(x)} \cdot \sin\frac{n\pi x}{L}\, dx \overset{\text{by parts}}{\underset{\text{once}}{=}} \frac{2L}{n\pi}(-1)^{n+1}$$

Therefore,

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L} \qquad (*)$$

Let's differentiate formally series (*) wrt x.

$$1 \overset{?}{\sim} \sum_{n=1}^{\infty} \frac{2L}{\cancel{n\pi}}(-1)^{n+1} \cdot \frac{\cancel{n\pi}}{\cancel{L}} \cos\frac{n\pi x}{L} = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos\frac{n\pi x}{L} \qquad (**)$$

But Fourier series of 1 is just 1:

$$1 = \underbrace{1}_{a_0} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad a_n = 0, \quad a_0 = 1$$

so series on the RHS of (**) is not a Fourier series of function 1.

Moreover, n^{th} term $2(-1)^{n+1}\cos\frac{n\pi x}{L} \not\to 0$ as $n \to \infty$ $\Rightarrow$ Fourier series on RHS of (**) does not even converge.

$\therefore$ Not every Fourier series can be differentiated term-by-term.