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Lecture 1

Math 480 : PDEs

Ch 1 INTRODUCTION

Def a differential equation (DE) is an equation that involves a derivative(s) and possibly a unknown function of at least one variable.

Ex $\frac{du}{dx} + 4u = \cos(4\pi x)$ $\qquad u = u(x)$

$y = ax + b$

This is an example of 1st order linear DE w/ constant coefficients. DE is nonhomogeneous.

Notation

t = time, $\quad t \in [0, +\infty)$

(x, y, z) : Cartesian spatial coordinates

(r, θ, z) : cylindrical spatial coordinates

(r, θ, φ) : spherical spatial coordinates

$u = u(t, x)$

$u_t = \dfrac{\partial u}{\partial t}$: 1st order partial derivative wrt t

$u_x = \dfrac{\partial u}{\partial x}$: — " — " — x

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$: 2nd order partial derivative wrt x

$u_{xt} = \dfrac{\partial^2 u}{\partial t \partial x}$: 2^{nd} order mixed partial derivative

Def A partial differential equation (PDE) is a DE that involves a function of at least two independent variables and derivatives wrt to at least two different variables.

Ex $u_t - u_{xx} = 0$ $\qquad u = u(t, x)$

Note In general, a solution (a function) of a PDE is not unique without prescribing initial condition(s) and/or boundary condition(s).

Roughly speaking,

highest order of time derivative = # of initial conditions

highest order of spatial derivative = # of boundary conditions

We will study 3 PDE from classical physics.

1. The heat / diffusion equation models distribution of heat or heat propagation in an object, evolution of concentration, e.g. evolution of pollutants etc.

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2. Laplace and Poisson equation used in electrostatics, ideal fluid flow, "steady state" heat/diffusion equation.

3. Wave equation used to model propagation of sound waves, seismic waves, gravity waves, electromagnetic waves etc.

We will study linear PDEs, most of them will be 1st or 2nd order.

Def The order of a PDE is the order of the highest derivative appearing in the equation. Sometimes distinction is made between variables.

Ex $u_t - u_{xx} = 0$

2nd order PDE

or we can say

1st order in t, and 2nd order in x

Ex $\dfrac{\partial^2}{\partial x^2}\left(EI \dfrac{\partial^2 u}{\partial x^2} \right) = -\mu \dfrac{\partial^2 u}{\partial t^2}$

E, I: functions of x

4th order PDE

4th order in x

2nd order in t

Def a PDE is <u>linear</u> if it can be written as a linear combination of $u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots$

<u>Aside:</u> $y = ax + b$: linear $f^{\underline{n}}$

linear n^{th} order ODE:

$$a_n(x)\frac{d^n u}{dx^n} + a_{n-1}(x)\frac{d^{n-1}u}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2u}{dx^2} +$$

$$+ a_1(x)\frac{du}{dx} + a_0(x)\,u = R(x)$$

A <u>second order linear</u> PDE can be written as:

$$\boxed{au + bu_t + cu_x + du_{xx} + eu_{xt} + fu_{tt} = g} \qquad (1)$$

where a, b, c, d, e, f, g may be functions of t and x (but not of u).

Def a <u>linear</u> PDE is <u>homogeneous</u> if $u \equiv 0$ is a solution of this equation. Otherwise, PDE is <u>nonhomogeneous</u>.

Note a 2^{nd} order linear PDE (1) is <u>homogeneous</u> if $g \equiv 0$. Otherwise, PDE is <u>nonhomogeneous.</u>

Ex $u_t - u_{xx} = \cos(xt)$: 2^{nd} order PDE, linear nonhomog.

Ex $\underbrace{x^2}_{=b} u_t - \underbrace{t^2}_{=c} u_x + \underbrace{xt}_{=a} \cdot u = \underbrace{3}_{=g}$: 1^{st} order PDE, linear nonhomog.

nonhomog.

Ex $u_t + \underbrace{u u_x}_{\text{nonlinearity}} = 0$: 1^{st} order, nonlinear

A DE can be written in an operator form

$$\mathcal{L}\{u\} = f$$

where $\mathcal{L}$ is a differential operator, u is an unknown function, f is a know function.

__Ex__ $\quad u_t - u_{xx} = \cos(xt) \qquad u = u(x,t)$

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \quad \Rightarrow \quad \mathcal{L}\{u\} = \cos(xt)$$

__Note__ a DE is <u>linear</u> if operator $\mathcal{L}$ does not involve u. Otherwise the DE is <u>nonlinear</u>.

Order of DE $=$ Order of $\mathcal{L}$

DE is homogeneous if $f \equiv 0$

<u>Please review the following</u>

<u>Review #1</u>: eigenvalues and eigenvectors

<u>Def</u> $Av = \lambda v$, then v and λ are called eigenvector – eigenvalue pair, $v \neq 0$, of matrix A.

e'values are solutions of the characteristic eq^{n}

$\det(A - \lambda I) = 0$

To find e'vectors, look for nontrivial ($v \neq 0$) solutions of $(A - \lambda I)v = 0$

<u>Note</u>: I expect you know how to find e'values & e'vectors.

Review #2 : Integration by parts

$$\int u\,dv = uv - \int v\,du$$

OR

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$