Ch 1 1.1. Introduction

Def A differential equation (DE) is an equation that involves a derivative(s) of an unknown function of at least one variable.

Ex $\frac{du}{dx} + 4u = \cos(4\pi x)$ $u = u(x)$

This is an example of an ordinary differential equation (ODE).

Notation

t = time, $t \in [0, +\infty)$

(x, y, z) : Cartesian spatial coordinates

(r, θ, z) : cylindrical spatial coordinates

(r, θ, φ) : spherical spatial coordinates

University of Idaho

$u = u(t, x)$

$u_t = \frac{\partial u}{\partial t}$: 1st order partial derivative wrt t

$u_x = \frac{\partial u}{\partial x}$: ——— // ——— wrt x

$u_{xx} = \frac{\partial^2 u}{\partial x^2}$: 2nd order partial derivative wrt x

$u_{xt} = \frac{\partial^2 u}{\partial t \partial x}$: 2nd order mixed partial derivative

Def a partial differential equation (PDE) is a DE that involves a function of at least two independent variables and derivatives wrt to at least two different variables.

Ex $u_t - u_{xx} = 0$ $\qquad u = u(t, x)$

Note In general, a solution (a function) of a PDE is not unique without prescribing initial conditions and/or boundary conditions.

Roughly speaking

highest order of time derivative = # of initial conditions

highest order of spatial derivative = # of boundary conditions

We will study 3 PDEs from classical physics.

1. The heat / diffusion equation models distribution of heat or heat propagation in an object, evolution of concentration, e.g. evolution of pollutants etc.

2. Laplace and Poisson equations used in electrostatics, ideal fluid flow, "steady state" heat / diffusion equations

3. Wave equation used to model propagation of sound, seismic waves, gravity waves, electromagnetic waves etc.

We will study linear PDEs, most of them will be 1st or 2nd order.

Def The order of a PDE is the order of the highest derivative appearing in the equation. Sometimes distinction is made between variables.

Ex $u_t - u_{xx} = 0$ — 2nd order PDE

or we can say
1st order in t, 2nd order in x

Ex $$\underbrace{\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u}{\partial x^2}\right) = -\mu \frac{\partial^2 u}{\partial t^2}}$$: 4th order or
4th order in x and
2nd order in t

EI: functions of x

Def a PDE is linear if it can be written as a linear combination of u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, ...

Aside:

$y = kx + b$: linear f^n

linear ODE of n^{th} order

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = R(x)$$

a 2^{nd} order linear PDE can be written as

$$\boxed{au + bu_t + cu_x + du_{tt} + eu_{tx} + fu_{xx} = g} \qquad (1)$$

where a, b, c, d, e, f, g may be functions of t and x (but not of u)

Def a linear PDE is homogeneous if $u \equiv 0$ is a solution of this equation.

a 2^{nd} order linear PDE (1) is homogeneous if $g \equiv 0$. Otherwise, PDE is nonhomogeneous.

Ex $u_t - u_{xx} = \cos(xt)$: 2nd order PDE, linear, nonhomog.

Ex $\underbrace{x^2}_{b} u_t - \underbrace{t^2}_{c} u_x + \underbrace{xt}_{a} \cdot u = 3$: 1st order, linear, nonhomog.

Ex $u_t + u u_x = 0$: 1st order, nonlinear

a DE can be written in an operator form

$$\mathcal{L}\{u\} = f$$

where $\mathcal{L}$ is a differential operator, u is unknown function, f is known function.

Ex $u_t - u_{xx} = \cos(xt)$ $\qquad u = u(t, x)$

$$\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \quad \Rightarrow \quad \mathcal{L}\{u\} = \cos(xt)$$

<u>Note</u> a DE is <u>linear</u> if operator $\mathcal{L}$ does not involve u. Otherwise the DE is <u>nonlinear</u>.

Order of DE = order of $\mathcal{L}$

DE is <u>homogeneous</u> if $f \equiv 0$

<u>Review #1</u>: eigenvalues and eigenvectors.

<u>Def</u> $Av = \lambda v$, then v and λ are called eigenvector – eigenvalue pair, $v \neq 0$, of matrix A.

e'values are solutions of characteristic equation

$$\det(A - \lambda I) = 0$$

I expect you know how to compute e'values & e'vectors.

<u>Review #2</u>: Integration by parts

$$\int u\,dv = uv - \int v\,du$$

or $\int f(x) g'(x)\,dx = f(x)g(x) - \int f'(x) g(x)\,dx$