Lecture 20

Consider <u>term-by-term differentiation of a Fourier sine series</u>

Let $$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where
$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \qquad (1)$$

Assume that $f(x)$ is continuous and $f'(x)$ is piecewise continuous.

If we differentiate formally term-by-term,

$$f'(x) \overset{?}{\sim} \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{L} \cos\frac{n\pi x}{L} \qquad (2)$$

Since $f'(x)$ is piecewise continuous, it has a Fourier cosine series

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad (3)$$

Note: if $f'(x)$ is continuous, then its Fourier cosine series converges to $f'(x)$. If $f'(x)$ has discontinuities, then Fourier cosine series will not converge to $f'(x)$.

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$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\, f(x)\Big|_{x=0}^{x=L} = \frac{1}{L}\left[f(L) - f(0)\right]$$

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx \overset{\text{by}}{\underset{\text{parts}}{=}} \left|\begin{array}{ll} u = \cos\frac{n\pi x}{L} & dv = f'(x)\,dx \\ du = -\frac{n\pi}{L}\sin\frac{n\pi x}{L} & v = f(x) \end{array}\right| =$$

$$= \frac{2}{L}\left\{\left(\cos\frac{n\pi x}{L}\cdot f(x)\right)\Big|_{x=0}^{x=L} + \frac{n\pi}{L}\underbrace{\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}_{\frac{L}{2}B_n}\right\} =$$

$$= \frac{2}{L}\left\{\underbrace{\cos n\pi}_{(-1)^n}\cdot f(L) - f(0)\underbrace{\cos 0}_{1} + \frac{n\pi}{\cancel{L}}\cdot\frac{\cancel{L}}{2}B_n\right\} =$$

$$= \frac{2}{L}\left\{(-1)^n f(L) - f(0) + \frac{n\pi}{2}B_n\right\}$$

$$\therefore \boxed{\begin{aligned} A_0 &= \frac{1}{L}\left[f(L) - f(0)\right] \\ A_n &= \frac{2}{L}\left[(-1)^n f(L) - f(0) + \frac{n\pi}{2} B_n\right], \quad n \geq 1 \end{aligned}}$$

Two Fourier cosine series (2) and (3) agree if

$$A_0 = 0 \quad \Rightarrow \quad \frac{1}{L}\left[f(L) - f(0)\right] = 0 \quad \Rightarrow \quad \boxed{f(L) = f(0)}$$

$$A_n = B_n \frac{n\pi}{L}$$

$$\Rightarrow \quad \frac{2}{L}\left[(-1)^n f(L) - f(0) + \cancel{\frac{n\pi}{2} B_n}\right] = \cancel{B_n \cdot \frac{n\pi}{L}}$$

$$\therefore \begin{cases} (-1)^n f(L) - f(0) = 0 \\ f(L) = f(0) \end{cases} \quad \Rightarrow \quad \boxed{f(L) = f(0) = 0}$$

$\therefore$ If $f'(x)$ is piecewise smooth, then Fourier sine series of a continuous function $f(x)$ <u>cannot</u> be differentiated term-by-term <u>in general</u>. Only when $f(L) = f(0) = 0$, then term-by-term differentiation is valid.

However, we showed that

$$f'(x) \sim \underbrace{\frac{1}{L}\left[f(L) - f(0)\right]}_{A_0} + \sum_{n=1}^{\infty} \frac{2}{L}\left[(-1)^n f(L) - f(0) + \frac{n\pi}{2} B_n\right] \cos\frac{n\pi x}{L} \qquad (4)$$

Ex Let's consider again Fourier sine series for $f(x)=x$

$$x \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad B_n = \frac{2L}{n\pi}(-1)^{n+1}$$

$f(x)=x$ $f(0)=0$ ✓ $f(L)=L\neq 0$

$f'(x)=1$: piecewise smooth function

Because $f(L)\neq 0$, term-by-term differentiation is <u>NOT</u> valid.

Let's use (4) to find correct expansion for $f'=1$:

$$1 \sim \frac{1}{L}[L-0] + \sum_{n=1}^{\infty} \frac{2}{L}\left[(-1)^n L - 0 + \frac{\cancel{n\pi}}{\cancel{L}}\cdot\frac{2L}{\cancel{n\pi}}(-1)^{n+1}\right]\cos\frac{n\pi x}{L}$$

or

$$1 \sim 1 + \sum_{n=1}^{\infty} \frac{2}{\cancel{L}}\left[(-1)^n \cancel{L} + \cancel{L}(-1)^{n+1}\right]\cos\frac{n\pi x}{L}$$

$$1 \sim 1 + \sum_{n=1}^{\infty} 2\left[\cancelto{0}{(-1)^n + (-1)^{n+1}}\right] \cos\frac{n\pi x}{L}$$

This is the correct Fourier cosine series for $f'(x)=1$.

Q What do these results about term-by-term differentiation mean for the heat equation problem w/ Dirichlet BCs?

- The only discontinuities occur at $t=0$.
- For all times $t>0$, the BCs are satisfied

$u(0,t)=0, \quad u(L,t)=0$

Homogeneous BCs allow us to differentiate Fourier sine series once to get Fourier cosine series, which can be differentiated term-by-term again without any extra BCs imposed. Any discontinuities present at $t=0$ (discontinuities in the IC $u(x,0) = f(x)$) will be smoothed out by $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$.

$\therefore$ We can differentiate term-by-term solution $u(x,t)$ of the heat problem w/ Dirichlet BCs.

Term-by-term integration of Fourier series

Thm The Fourier series of a piecewise smooth function $f(x)$ can ALWAYS be integrated term-by-term, and the resulting series is a convergent infinite series that always converges to the integral of $f(x)$ in $-L \le x \le L$ (even if the original Fourier series of $f(x)$ had jump discontinuities).

<u>Differentiation</u>: decreases smoothness

<u>Integration</u>: increases smoothness

<u>Aside</u>:

$f(x) = x^{1/2}$

$f(x) = x^{3/2}$

let

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$-L \le x \le L$$

we assume that $f(x)$ is piecewise smooth.

Integrate $\int_{-L}^{x}$

Define

$$F(x) \overset{def}{\equiv} \int_{-L}^{x} f(t)\,dt \qquad F'(x) = \frac{d}{dx}\int_{-L}^{x} f(t)\,dt = f(x)$$

$$\int_{-L}^{x} \cos\frac{n\pi t}{L}\,dt = \frac{L}{n\pi}\sin\frac{n\pi t}{L}\Big|_{t=-L}^{t=x} = \frac{L}{n\pi}\left(\sin\frac{n\pi x}{L} - \underbrace{\sin\frac{n\pi(-L)}{L}}_{=\sin(-n\pi)\to 0}\right)$$

$$= \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

a_0: $\int_{-L}^{x} 1\,dt = t\Big|_{t=-L}^{t=x} = x+L$

$$\int_{-L}^{x} \sin\frac{n\pi t}{L}\,dt = -\frac{L}{n\pi}\cos\frac{n\pi t}{L}\Big|_{t=-L}^{t=x} =$$

$$= -\frac{L}{n\pi}\Big[\cos\frac{n\pi x}{L} - \underbrace{\cos\frac{n\pi(-L)}{L}}_{=\cos(-n\pi) \overset{\text{even}}{\underset{f^n}{=}} \cos n\pi = (-1)^n}\Big] = -\frac{L}{n\pi}\Big[\cos\frac{n\pi x}{L} - (-1)^n\Big]$$

We would like to show that

$$F(x) \overset{?}{=} a_0(x+L) + \sum_{n=1}^{\infty} a_n \frac{L}{n\pi}\sin\frac{n\pi x}{L} +$$

$$+ \sum_{n=1}^{\infty} b_n\left(-\frac{L}{n\pi}\right)\Big[\cos\frac{n\pi x}{L} - (-1)^n\Big]$$

OR

$$F(x) \overset{?}{=} a_0(x+L) + \sum_{n=1}^{\infty} a_n\frac{L}{n\pi}\sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\Big[(-1)^n - \cos\frac{n\pi x}{L}\Big] \qquad (\Delta)$$