Lecture 20

Theorem

a Fourier series that is continuous (has no jump discontinuities) can be differentiated term-by-term if $f'(x)$ is piecewise smooth.

alternatively, we can write explicitly what continuity of Fourier series means.

Thm 4 If $f'(x)$ is piecewise smooth, then Fourier series of a continuous function $f(x)$ can be differentiated term-by-term if $f(-L) = f(L)$.

Thm 5 If $f'(x)$ is piecewise smooth, then Fourier cosine series of a continuous function $f(x)$ can be differentiated term-by-term.

Thm 6 If $f'(x)$ is piecewise smooth, then Fourier sine series of a continuous function $f(x)$ can be differentiated term-by-term if $f(0) = f(L) = 0$.

Note Jump discontinuities in the domain or at end points are the reason that causes term-by-term differentiation to be invalid.

Consider term-by-term differentiation of a Fourier sine series.

Let $$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where
$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad (1)$$

Assume that $f(x)$ is continuous and $f'(x)$ is piecewise continuous.

If we differentiate formally term-by-term,

$$f'(x) \overset{?}{\sim} \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{L} \cos \frac{n\pi x}{L} \qquad (2)$$

Since $f'(x)$ is piecewise continuous, it has a Fourier cosine series

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \qquad (3)$$

<u>Note</u> If $f'(x)$ is continuous, then its Fourier cosine series converges to $f'(x)$. If $f'(x)$ has discontinuities, then Fourier cosine series will not converge to $f'(x)$.

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$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\left[f(L) - f(0)\right]$$

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx \overset{\text{by}}{\underset{\text{parts}}{=}} \left|\begin{array}{ll} u = \cos\frac{n\pi x}{L} & dv = f'(x)\,dx \\ du = -\frac{n\pi}{L}\sin\frac{n\pi x}{L}\,dx & v = f(x) \end{array}\right|$$

$$= \frac{2}{L}\left[f(x)\cos\frac{n\pi x}{L}\Big|_{x=0}^{x=L} + \frac{n\pi}{L}\underbrace{\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}_{\frac{L}{2}B_n}\right] =$$

$$= \frac{2}{L}\left[f(L)\underbrace{\cos n\pi}_{(-1)^n} - f(0)\underbrace{\cos 0}_{=1} + \frac{n\pi}{\cancel{L}}\cdot\frac{\cancel{L}}{2}B_n\right] =$$

$$= \frac{2}{L}\left[f(L)(-1)^n - f(0) + \frac{n\pi}{2}B_n\right]$$

$$\therefore \quad \boxed{\begin{array}{l} A_0 = \frac{1}{L}\left[f(L) - f(0)\right] \\ A_n = \frac{2}{L}\left[f(L)(-1)^n - f(0) + \frac{n\pi}{2}B_n\right], \quad n \geq 1 \end{array}}$$

Two Fourier series (2) and (3) agree if

$$A_0 = 0 \quad \Rightarrow \quad \frac{1}{L}\left[f(L) - f(0) \right] = 0 \quad \Rightarrow \quad \boxed{f(L) = f(0)}$$

$$A_n = B_n \cdot \frac{n\pi}{L}$$

$$\therefore \quad \underline{\underline{B_n \cdot \frac{n\pi}{L}}} = \frac{2}{L}\left[f(L)(-1)^n - f(0) + \underline{\underline{\frac{n\pi}{2} B_n}} \right]$$

$$\left.\begin{aligned} &\Rightarrow f(L)(-1)^n - f(0) = 0 \\ &\text{But} \quad f(L) = f(0) \end{aligned}\right\} \Rightarrow \boxed{f(L) = f(0) = 0}$$

$\therefore$ If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ cannot be differentiated term-by-term in general. Only when $f(L) = f(0) = 0$, the term-by-term differentiation is valid.

However, we showed that

$$\boxed{f'(x) \sim \underbrace{\frac{1}{L}\left[f(L) - f(0)\right]}_{A_0} + \sum_{n=1}^{\infty} \underbrace{\frac{2}{L}\left[f(L)(-1)^n - f(0) + \frac{n\pi}{2} B_n\right]}_{A_n} \cos\frac{n\pi x}{L}} \qquad (4)$$

Ex Let's consider again Fourier sine series for $f(x) = x$.

$$x \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad B_n = \frac{2L}{n\pi}(-1)^{n+1}$$

$f(x) = x$, $\quad f(0) = 0$ ✓ $\qquad f(L) = L \neq 0$

$f'(x) = 1$: piecewise smooth function!

Formal term-by-term differentiation is not valid here.

Let's use formula (4):

$$1 \sim \frac{1}{L}\left[L - 0\right] + \sum_{n=1}^{\infty} \frac{2}{L}\left[L(-1)^n - 0 + \frac{\cancel{n\pi}}{\cancel{2}} \cdot \frac{\cancel{2}L}{\cancel{n\pi}}(-1)^{n+1}\right] \cos\frac{n\pi x}{L}$$

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or

$$1 \sim 1 + \sum_{n=1}^{\infty} 2\left[(-1)^n \cancelto{0}{+(-1)^{n+1}}\right] \cos\frac{n\pi x}{L} = 1$$

This is the correct Fourier cosine series of $f'(x)=1$.

<u>Q</u> What do these results about term-by-term differentiation mean for the heat equation with Dirichlet BCs?

- The only discontinuities occur at $t=0$.
- For all times $t>0$, the BCs are satisfied, $u(0,t)=0$, $u(L,t)=0$.

Homogeneous BCs allow us to differentiate Fourier series once to get Fourier cosine series, which can be differentiated again without any extra BCs imposed. Any discontinuities present at $t=0$ (discontinuities in IC $u(x,0)=f(x)$ will be smoothed out by $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$.

$\therefore$ We can differentiate term-by-term solution $u(x,t)$ of the heat problem w/ Dirichlet BCs.

Term-by-term integration of Fourier series

<u>Thm</u> The Fourier series of a piecewise smooth function $f(x)$ can ALWAYS be integrated term-by-term, and the resulting series is a convergent infinite series that always converges to the integral of $f(x)$ in $-L \leq x \leq L$ (even if the original Fourier series of $f(x)$ had jump discontinuities).

<u>Differentiation</u> : decreases smoothness

<u>Integration</u> : increases smoothness

Aside

$f(x) = x^{1/2}$

$f(x) = x^{3/2}$