Lecture 21

Term-by-term integration of Fourier series (Cont'd)

Let $f(x)$ be piecewise smooth and

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Formally integrate both sides $\int_{-L}^{x} \ldots$

Introduce

$$F(x) = \int_{-L}^{x} f(t)\,dt$$

We would like to show (see previous lecture):

$$F(x) \overset{?}{=} a_0(x+L) + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \quad (\Delta)$$

Define $\boxed{G(x) = F(x) - a_0(x+L)}$

$$F(x) = \int_{-L}^{x} f(t)\,dt$$

$$F(-L) = \int_{-L}^{-L} f(t)\,dt = 0$$

$$F(L) = \int_{-L}^{L} f(t)\,dt$$

$$G(-L) = \cancelto{0}{F(-L)} - a_0\,\cancelto{0}{(-L+L)} = 0$$

$$G(L) = F(L) - a_0(L+L) = F(L) - 2La_0 = 0$$

Recall $a_0 = \dfrac{1}{2L} \underbrace{\int_{-L}^{L} f(t)\,dt}_{F(L)}$

Since function $f(x)$ is piecewise smooth, the antiderivative $F(x)$ of $f(x)$ is continuous. Function $a_0(x+L)$ is linear, so it is also continuous. Hence, $G(x) = F(x) - a_0(x+L)$ is also continuous. In addition, $G(-L) = G(L) = 0$.

This implies that $G(x)$ has a continuous Fourier series that converges to $G(x)$ on $[-L, L]$.

$$G(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

note equality sign

Recall $G(x) = F(x) - a_0(x+L)$. Let's compute coefficients A_n's and B_n's.

$$A_n = \frac{1}{L}\int_{-L}^{L} \left[F(x) - a_0(x+L)\right] \cos\frac{n\pi x}{L}\,dx \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$u = F(x) - a_0(x+L) \qquad F(x) = \int_{-L}^{x} f(t)\,dt$$

$$du = (F'(x) - a_0)\,dx = (f(x) - a_0)\,dx \qquad F'(x) = f(x)$$

$$dv = \cos\frac{n\pi x}{L}\,dx \qquad v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$= \frac{1}{L}\left\{ \left(F(x) - a_0(x+L)\right)\frac{L}{n\pi}\cancelto{0}{\sin\frac{n\pi x}{L}}\Bigg|_{x=-L}^{x=L} - \int_{-L}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,(f(x)-a_0)\,dx \right\}$$

$$= -\frac{1}{\cancel{L}}\cdot\frac{\cancel{L}}{n\pi}\Bigg[\underbrace{\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}_{=\,L\cdot b_n} - a_0\cancelto{0}{\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx}\Bigg\} = -\frac{1}{n\pi}\,b_n L$$

odd f^n over symmetric interval

$$\therefore \quad \boxed{A_n = -\frac{b_n L}{n\pi}} \qquad (1)$$

Here we used the fact that $f(x)$ has a Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Similarly, we can show

$$\boxed{B_n = \frac{a_n L}{n\pi}} \qquad (2)$$

$\Rightarrow$

$$G(x) = A_0 + \sum_{n=1}^{\infty} \underbrace{A_n}_{= -\frac{b_n L}{n\pi}} \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \underbrace{B_n}_{= \frac{a_n L}{n\pi}} \sin\frac{n\pi x}{L} \qquad (3)$$

or

$$G(x) = A_0 + \sum_{n=1}^{\infty}\left(-\frac{b_n L}{n\pi}\right)\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\frac{a_n L}{n\pi}\sin\frac{n\pi x}{L} \qquad (4)$$

$$0 = G(L) \overset{(3)}{=} A_0 + \sum_{n=1}^{\infty} A_n \underbrace{\cos(n\pi)}_{(-1)^n} + \sum_{n=1}^{\infty} B_n \cancel{\sin(n\pi)} \xrightarrow{} 0$$

$$\therefore \quad A_0 = -\sum_{n=1}^{\infty} \underbrace{A_n}_{= -\frac{b_n L}{n\pi}} (-1)^n = \boxed{\sum_{n=1}^{\infty} \frac{b_n L}{n\pi}(-1)^n = A_0} \qquad (5)$$

Substituting (5) into (4), we get

$$G(x) = \sum_{n=1}^{\infty}\left(\frac{b_n L}{n\pi}(-1)^n - \frac{b_n L}{n\pi}\cos\frac{n\pi x}{L} + \frac{a_n L}{n\pi}\sin\frac{n\pi x}{L}\right)$$

$\shortparallel$

$F(x) - a_0(x+L)$

$$F(x) - a_0(x+L) = \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \qquad (*)$$

or

$$F(x) = a_0(x+L) + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \qquad (**)$$

Compare this result w/ (Δ) : they are the same!

<u>Note</u> that series on the RHS of (**) is NOT a Fourier series because of the linear term $a_0 x$. But the series on the RHS of (*) is a convergent continuous Fourier series of $F(x) - a_0(x+L)$.

Complex Form of Fourier Series

let

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Recall

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \text{ Euler's identities}$$

Then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$i = \sqrt{-1} \quad \text{or} \quad i^2 = -1$$

Then

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} =$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} =$$

$$\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

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$$\circledcirc\; a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - i b_n)e^{i\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + i b_n)e^{-i\frac{n\pi x}{L}} =$$

$n \to -n$

$$= \underset{= c_0}{a_0} + \frac{1}{2}\sum_{n=-1}^{-\infty}\underbrace{(a_{-n} - i b_{-n})}_{\substack{a_n + i b_n \\ c_n}} e^{-i\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}\underbrace{(a_n + i b_n)}_{c_n}e^{-i\frac{n\pi x}{L}}$$

From definitions of a_n, b_n, we can write

$a_{-n} = a_n$ and $b_{-n} = -b_n$

Denote $c_0 = a_0$

$c_n = \frac{1}{2}(a_n + i b_n)$

$$\therefore \quad \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}}$$

complex form of a Fourier series

$$c_n = \frac{1}{2}(a_n + i b_n) = \frac{1}{2L}\int_{-L}^{L} f(x)\,\underbrace{\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)}_{e^{i\frac{n\pi x}{L}}}\,dx =$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi x}{L}}\,dx$$

$$\therefore \quad \boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi x}{L}}\,dx}$$