Lecture 21

Term-by-term integration of Fourier series (Cont'd)

Let

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}, \qquad -L \le x \le L$$

we assume that $f(x)$ is piecewise smooth

Integrate $\int_{-L}^{x}$

$$F(x) \overset{def}{\equiv} \int_{-L}^{x} f(t)\,dt \qquad F'(x) = \frac{d}{dx}\int_{-L}^{x} f(t)\,dt = f(x)$$

$$\int_{-L}^{x} \cos\frac{n\pi t}{L}\,dt = \frac{L}{n\pi}\sin\frac{n\pi t}{L}\Big|_{t=-L}^{t=x} = \frac{L}{n\pi}\left(\sin\frac{n\pi x}{L} - \underbrace{\sin(-n\pi)}_{0}\right) = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$\int_{-L}^{x} \sin\frac{n\pi t}{L}\,dt = -\frac{L}{n\pi}\cos\frac{n\pi t}{L}\Big|_{t=-L}^{t=x} = -\frac{L}{n\pi}\Big[\cos\frac{n\pi x}{L} - \underbrace{\cos(-n\pi)}_{\cos(n\pi)=(-1)^n}\Big] =$$

$$= -\frac{L}{n\pi}\Big[\cos\frac{n\pi x}{L} - (-1)^n\Big]$$

We would like to show that

$$F(x) \stackrel{?}{=} a_0(x+L) + \sum_{n=1}^{\infty} a_n\frac{L}{n\pi}\sin\frac{n\pi x}{L} + b_n\left(-\frac{L}{n\pi}\right)\Big[\cos\frac{n\pi x}{L} - (-1)^n\Big]$$

or

$$F(x) \stackrel{?}{=} a_0(x+L) + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi}\sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\Big[(-1)^n - \cos\frac{n\pi x}{L}\Big] \qquad (\Delta)$$

Define $\boxed{G(x) = F(x) - a_0(x+L)}$

$$G(-L) = \cancelto{0}{F(-L)} - a_0(\cancelto{0}{-L+L}) = 0$$

since $F(x) = \int_{-L}^{x} f(t)\,dt$

$$F(-L) = \int_{-L}^{-L} f(t)\,dt$$

$$G(L) = F(L) - a_0\cdot 2L = 0$$

since

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \qquad F(L) = \int_{-L}^{L} f(t)\,dt$$

Since $f(x)$ is piecewise smooth $\Rightarrow$ antiderivative $F(x)$ of $f(x)$ is continuous. Function $a_0(x+L)$ is linear, so it is also continuous $\Rightarrow$ $G(x) = F(x) - a_0(x+L)$ is also continuous. In addition $G(-L) = G(L) = 0$. This implies that $G(x)$ has a continuous Fourier series that converges to $G(x)$ on $[-L, L]$.

$$\therefore \quad G(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

note equality sign

Recall

$$G(x) = F(x) - a_0(x+L)$$

$$A_n = \frac{1}{L}\int_{-L}^{L}\left[F(x) - a_0(x+L)\right]\cos\frac{n\pi x}{L}\,dx \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$= \left|\begin{array}{ll} u = F(x) - a_0(x+L) & dv = \cos\frac{n\pi x}{L}\,dx \\ du = u'(x)\,dx & v = \frac{L}{n\pi}\sin\frac{n\pi x}{L} \\ du = (f(x) - a_0)\,dx & \end{array}\right| = \begin{array}{l} u' = F' - a_0 = f(x) - a_0 \\ F' = \frac{d}{dx}\int_{-L}^{x} f(t)\,dt = f(x) \end{array}$$

$$= \frac{1}{L}\left\{\left[F(x) - a_0(x+L)\right]\cdot\frac{L}{n\pi}\cancelto{0}{\sin\frac{n\pi x}{L}}\Bigg|_{x=-L}^{x=L} - \frac{L}{n\pi}\int_{-L}^{L}\left(f(x) - a_0\right)\cdot\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= -\frac{1}{\cancel{L}}\cdot\frac{\cancel{L}}{n\pi}\left[\underbrace{\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}_{b_n\cdot L} - a_0\int_{-L}^{L}\cancelto{0}{\sin\frac{n\pi x}{L}}\,dx\right] = -\frac{1}{n\pi}\cdot b_n\cdot L$$

odd f^{n} over symmetric interval

$$\Rightarrow \boxed{A_n = -\frac{b_n\cdot L}{n\pi}} \qquad (1)$$

since Fourier coefficients of Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

similarly, we can show $\boxed{B_n = \dfrac{a_n L}{n\pi}}$ (2)

$$G(x) = A_0 + \sum_{n=1}^{\infty} \underbrace{A_n}_{= -\frac{b_n L}{n\pi}} \cos\frac{n\pi x}{L} + \underbrace{B_n}_{= \frac{a_n L}{n\pi}} \sin\frac{n\pi x}{L}$$

$$0 = G(L) = A_0 + \sum_{n=1}^{\infty} A_n (-1)^n \Rightarrow A_0 = -\sum_{n=1}^{\infty} A_n (-1)^n \quad \text{since } A_n = -\frac{b_n L}{n\pi}$$

$$\boxed{A_0 = \sum_{n=1}^{\infty} \frac{b_n L}{n\pi} (-1)^n}$$

$$\Rightarrow G(x) = \sum_{n=1}^{\infty} \left(\underbrace{\frac{b_n L}{n\pi}(-1)^n - \frac{b_n L}{n\pi} \cos\frac{n\pi x}{L}} + \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} \right)$$

$$\parallel$$

$$F(x) - a_0(x+L)$$

$$\Rightarrow F(x) - a_0(x+L) = \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \quad (*)$$

or

$$F(x) = a_0(x+L) + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \quad (**)$$

Compare this result with (Δ) : the same!

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Note that series on the RHS of (**) is NOT a Fourier series because of the linear term $a_0 \cdot x$. But the series on the RHS of (*) is a convergent continuous Fourier series of $F(x) - a_0(x+L)$.

Complex Form of Fourier Series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Recall

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Euler's identities

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$i = \sqrt{-1}$ or $i^2 = -1$

Then

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cdot \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} +$$

$$+ \sum_{n=1}^{\infty} b_n \cdot \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} =$$

$$\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n - i b_n) e^{i\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n + i b_n) e^{-i\frac{n\pi x}{L}} =$$

$n \to -n$

$$= a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty} (\underbrace{a_{-n}}_{=a_n} - i\underbrace{b_{-n}}_{=-b_n}) e^{-i\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + i b_n) e^{-i\frac{n\pi y}{L}}$$

From definition of a_n, b_n, we can see that

$$a_{-n} = a_n \quad \text{and} \quad b_{-n} = -b_n$$

Denote

$$c_0 = a_0$$

$$c_n = (a_n + i b_n)\cdot\frac{1}{2}$$

$$\therefore \boxed{f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}}$$

<u>complex form of Fourier series</u>

$$c_n = (a_n + i b_n)\cdot\frac{1}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\Big(\underbrace{\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}}_{=e^{i\frac{n\pi x}{L}}}\Big)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}}\,dx$$

$$\therefore \boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}}\,dx}$$