Lecture 22

Complex Orthogonality

Recall

Def Two real-valued functions $\psi(x)$ and $\phi(x)$ defined on $a \le x \le b$ are orthogonal on $a \le x \le b$ if

$$\int_a^b \psi(x)\,\phi(x)\,dx = 0$$

$$\langle \psi, \phi \rangle = \int_a^b \psi(x)\,\phi(x)\,dx$$

inner product of ψ & ϕ on $[a, b]$

Def Two complex-valued functions $\psi(x)$ and $\phi(x)$ are orthogonal on $[a, b]$ if

$$\int_a^b \overline{\Psi}(x)\, \Phi(x)\, dx = 0 \qquad \langle \Psi, \Phi \rangle = \int_a^b \overline{\Psi}\, \Phi\, dx$$

where $\overline{\Psi}(x)$ is a complex conjugate of $\Psi(x)$.

Here $\Psi(x) = f(x) + i\,g(x)$

$\overline{\Psi(x)} = f(x) - i\,g(x)$

$f(x) = \mathrm{Re}\,\Psi(x)$: real part of $\Psi(x)$

$g(x) = \mathrm{Im}\,\Psi(x)$: imaginary part of $\Psi(x)$

Note: $f(x), g(x)$ are real-valued functions

Complex numbers:

$z = x + iy$: complex #

$x = \mathrm{Re}\, z, \quad y = \mathrm{Im}\, z$

x, y: real #s

$\overline{z} = x - iy$: complex conjugate of z

<u>Note</u> $\overline{e^{-i\frac{n\pi x}{L}}} = e^{i\frac{n\pi x}{L}}$

$$\begin{cases} e^{it} = \cos t + i\sin t \\ e^{-it} = \cos t - i\sin t \\ \overline{e^{-it}} = e^{it} \end{cases}$$

<u>Claim</u> Functions $e^{-i\frac{n\pi x}{L}}$, $n=-\infty,\dots,+\infty$ are orthogonal on $[-L,L]$.

□ $$\langle e^{-i\frac{m\pi x}{L}}, e^{-i\frac{n\pi x}{L}} \rangle = \int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}} \cdot e^{-i\frac{n\pi x}{L}}\, dx =$$

$$= \int_{-L}^{L} e^{i\frac{m\pi x}{L}} e^{-i\frac{n\pi x}{L}}\, dx = \int_{-L}^{L} e^{i\frac{(m-n)\pi x}{L}}\, dx = \begin{cases} 2L, & m=n \\ [\,..\,], & m\neq n \end{cases}$$

$m \neq n$

$$[\dots] = \int_{-L}^{L} e^{i\frac{(m-n)\pi x}{L}}\,dx = \frac{L}{(m-n)\pi i}\, e^{i\frac{(m-n)\pi x}{L}}\Bigg|_{x=-L}^{x=L} =$$

$$= \frac{L}{(m-n)\pi i}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right] =$$

$$= \frac{L}{(m-n)\pi i}\cdot 2i \sin \xrightarrow{0}\cancel{(m-n)\pi} = 0$$

$$e^{it} - e^{-it} = 2i\sin t$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$

$$e^{it} - e^{-it} \overset{\text{Euler}}{\underset{\text{ident}}{=}} \cancel{\cos t} + i\sin t - (\cancel{\cos t} - i\sin t) = 2i\sin t$$

$\therefore$

$$\boxed{\left\langle e^{-i\frac{m\pi x}{L}}, e^{-i\frac{n\pi x}{L}}\right\rangle = \int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}}\, e^{-i\frac{n\pi x}{L}}\,dx = \begin{cases} 2L, & m = n \\ 0, & m \neq n \end{cases}}$$

Assume

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}} \qquad \Big| \ \overline{e^{-i\frac{m\pi x}{L}}} \qquad \int_{-L}^{L}$$

$$\int_{-L}^{L} f(x)\,\overline{e^{-i\frac{m\pi x}{L}}}\,dx \sim \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-L}^{L} e^{-i\frac{n\pi x}{L}}\,\overline{e^{-i\frac{m\pi x}{L}}}\,dx}_{= \begin{cases} 2L, & m = n \\ 0, & m \neq n \end{cases}}$$

$$\therefore \ \boxed{c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{i\frac{m\pi x}{L}}\,dx \qquad \text{for all } m}$$

we can change $m \to n$.

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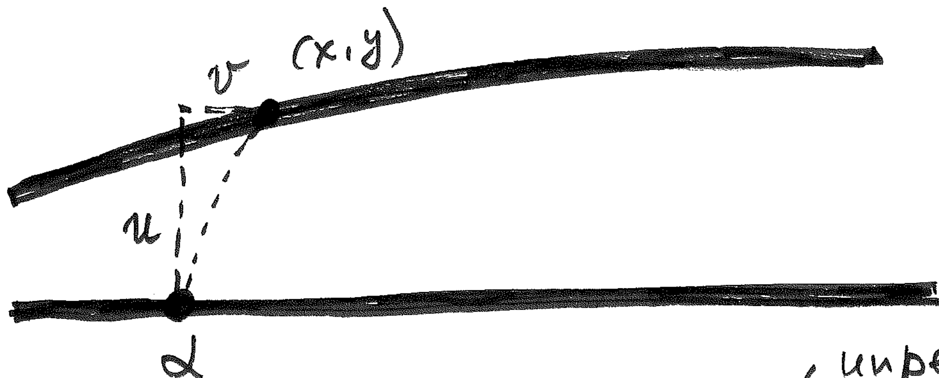
Note this is the same formula for C_m (or C_n) as one we derived last time (see lecture 21).

Note Complex form of a Fourier series is a more compact form and it is more convenient to use.

Chapter 4 Wave Equation



displaced / perturbed string

unperturbed string (highly stretched)

equilibrium state

u: vertical displacement of particle α

v: horizontal displacement of α, $v \ll u$

Let α be the x-coordinate of a particle / point sitting on a string in equilibrium position where string is highly stretched.

Now, let us perturb the string. Denote by u and v vertical and horizontal displacements of the particle. Assume that the slope of the perturbed string is small $\Rightarrow$ we can neglect horizontal displacements and motion becomes completely vertical.

$$y = u(x,t) \quad \text{where} \quad x = \alpha$$

$$\therefore \quad (v \approx 0, \ x \approx \alpha)$$

Goal: derive governing equation (PDE) for $u(x,t)$.

We will use Newton's 2^{nd} law of motion $\vec{F} = m\vec{a}$ and consider vertical component.

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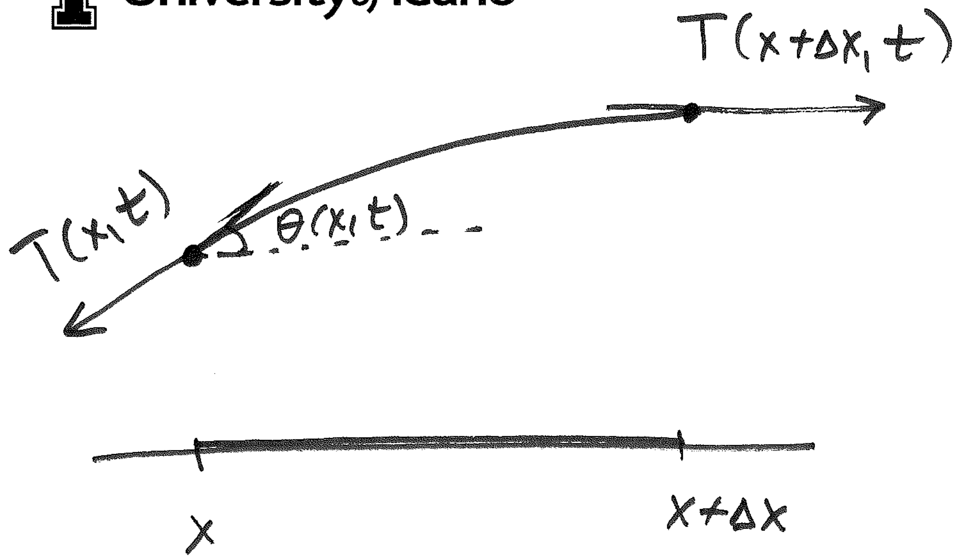
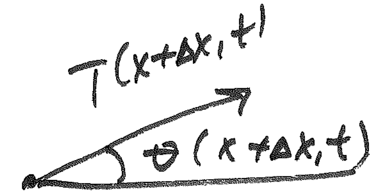


Tensile force tries to stretch the string segment



Assume that mass density $\rho_0(x)$ is known. Then the total mass of the ~~segment~~ $[x, x+\Delta x]$ is $\rho_0(x)\,\Delta x$. We assume that string is <u>perfectly flexible</u>, i.e. there is no resistance to bending $\Rightarrow$ the rest of the string exerts the force on end points that act in the direction tangent to the string. This force is <u>tension</u> of the string with magnitude $T(x,t)$. The other force is the body force $Q(x,t)$ (gravity).

Let $\theta(x,t)$ be the angle between positive x-axis and tangent line to the string.

$$y = u(x,t)$$

Slope of the string $= \dfrac{dy}{dx} = \tan\theta = \dfrac{\partial u}{\partial x}$

Newton's 2nd law (vertical component) $m\vec{a} = \vec{F}$

gives

$$\rho_0(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)$$

$$+\ \rho_0(x)\,\Delta x\,\underbrace{Q(x,t)}_{\text{vertical component of gravity per unit mass}}$$

Divide both sides by Δx and let $\Delta x \to 0$.

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} +$$

$$+ \rho_0(x)\, Q(x,t)$$

as $\Delta x \to 0$, we get

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\sin\theta(x,t)\Big) + \rho_0(x)\, Q(x,t)$$

Since θ is a small angle, we have:

θ : small

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta \qquad \Rightarrow \begin{array}{l} \sin\theta \simeq \theta \\ \cos\theta \approx 1 \end{array}$$

Hence,

$$\boxed{\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t) \frac{\partial u}{\partial x}\right) + \rho_0(x)\, Q(x,t)}$$